Application of Taylor Expansion on Calculating Functions

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Abstract. The Taylor formula, which approximates some complicated functions as straightforward polynomial functions, is a crucial concept in advanced mathematics. It is an effective tool for studying and analyzing various mathematical topics because of its ability to reduce complexity. The "approximation method" of calculus is embodied in the Taylor formula, which also offers special advantages for approximation calculations. Taylor's formula has significant applications in all areas of calculus because it can accurately convert nonlinear issues into linear problems. The Taylor formula can be used to determine a number of things, including a limit, a function's extreme value, the value of the higher derivative at a certain point, the convergence of the generalized integral, an approximation, the proving of an inequality, and more. This paper primarily presents Taylor's expansion formula's significance and provides concrete examples of how to apply it to solve difficulties. The detailed writing steps to understand the problem are shown as well.

Keywords: Taylor expansion formula; Definite integral; Approximate method; Real analysis.

1. Introduction

The Taylor formula is a powerful tool in contemporary mathematics. Mathematician Brook Taylor, one of the best Newtonian school representatives in the early 18th century, published his main work "Positive and inverse incremental methods" in 1715. In this work, he established the well-known theorem, known as Taylor's theorem, which he initially proposed to his teacher Machin in a letter in July 1712. Taylor used the Taylor theorem to resolve numerical problems in 1717. The Gregorian-Newton interpolation formula, which is a formula that provides information to characterize the value of a function near a specific point, is the basis for Taylor's formula. Given the derivatives of the function at a particular location, if the function is sufficiently homogeneous, Taylor's formula can employ these component values as factors to produce a smoother function. Taylor's formula can use these differential values as coefficients to come up with a polynomial that closely approximates the function's value at that location in the surrounding area. In 1772, Lagrange underlined the critical nature of Taylor's formula, referring to it as the basic axiom of the calculus of differential equations. However, Cauchy's work in the 1820s was the first to incorporate the convergence of series into the proof of Taylor's theorem. Since any function of a single variable may now be stretched into a power series thanks to Taylor's theorem, he is regarded as the creator of finite difference theory [1]. Taylor's formula has significant applications in all areas of calculus because it can accurately convert nonlinear issues into linear problems. The Taylor formula can be used to determine a number of things, including a limit, a function's extreme value, the value of the higher derivative at a certain point, the convergence of the generalized integral, an approximation, the proving of an inequality, and more [2].

The Harris corner detection principle can be used in the Taylor expansion formula. The human eye's diagonal point is typically recognized in a localized little space or tiny window. There is no corner in the window if the small window of this feature is moved in all directions but the gray level does not change. If the window is moved in one direction but the gray level significantly changes and the other direction does not change, the image in the window may be a straight-line segment. First of all, the image window should be translated to produce an auto correlation function of gray level change, and then the expression of this can be solved to know whether the corner is encountered in the current window, and the solution of this function needs to use the Taylor expansion formula [3].

This paper will look at four aspects of Taylor's formula. In the first part, this paper will introduce some historical background of Taylor's formula, and will write out the importance of Taylor's expansion formula. In the second part, this paper will first introduce the significance of Taylor's formula, and then introduce three different types of Taylor's expansion formula, and further study. In the third part, this paper will introduce the application of Taylor's formula in the problems, respectively listed three different types of problems, and explain these problems. In the fourth part, this paper will summarize some methods of Taylor's expansion formula and put references at the end.

2. Fundamental Theorys

2.1. Definition of Taylor Expansion

The Taylor Expansion formula employs details from a function's values at an agreed-upon location to describe a function's value adjacent. If a number of criteria are met, an approximation of Taylor Expansion can approximate the function by constructing a polynomial with coefficients equal to the coefficients of the variable's derivatives at a given location.

If a function known as $f(x)$ that has an n-th derivative on a closed interval [a, b] containing x_0, and an $(n+1)$-th derivative on the open portion of the interval (a, b). Thus, with respect to the closed interval $[a,b]$ at any point x above, there is [4]

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}+R_n(x). \tag{1}$$

Here, $f^{(n)}(x_0)$ indicates that $f(x)$ has a derivative of order n on some closed interval $[a,b]$, containing x. The polynomial after the equal sign is called the Taylor Expansion of the function $f(x)$ at x_0 , and the remainder $R_n(x)$ is the residue of Taylor expansion formula and is an infinitely small quantity of higher order of $(x-x_0)^n$.

2.2. Types of Taylor Expansion

One of the types is the Taylor expansion formula with Peano residue. If the function $f(x)$ has an n derivative at x_0, then $f(x)$ can be expanded to [5]

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+0(x-x_0)^n). \tag{2}$$

This type can calculate the asymptote of the function. This ideal is use $\mathrm{g}\left(\frac{1}{\mathrm{x}}\right)=\frac{\mathrm{f(x)}}{\mathrm{x}}$, then $\lim_{x\to\infty} g\left(\frac{1}{x}\right)=k$. Define $g(0)=k$, then $g(x)$ continue at $x=0$. If $g'(0^+)$ exist, then $g'(0^+)=\lim_{x\to\infty^+}\left[xg\left(\frac{1}{x}\right)-kx\right]=b$, so $g'(0^+)$ exist and equal to b, then $g(t)$ can be expanded at 0, have $\mathrm{g}\left(\frac{1}{\mathrm{x}}\right)=\mathrm{k}+\frac{\mathrm{g}'(0^+)}{\mathrm{x}}+\mathrm{o}\left(\frac{1}{\mathrm{x}}\right)$. Thus, one has [6]

$$\mathrm{x}\frac{\mathrm{f(x)}}{\mathrm{x}}=\mathrm{xg}\left(\frac{1}{\mathrm{x}}\right)=\mathrm{x}\left[\mathrm{k}+\frac{\mathrm{g}'(0^+)}{\mathrm{x}}+\mathrm{o}\left(\frac{1}{\mathrm{x}}\right)\right]=\mathrm{kx}+\mathrm{g}'(0^+)+\mathrm{xo}\left(\frac{1}{\mathrm{x}}\right) \tag{3}$$

So, $\mathrm{y}=\mathrm{kx}+\mathrm{g}'(0^+)$, and it is an asymptote of $f(x)$.

The Taylor expansion formula with Lagrange residue is another variant. Taylor Expansion formula with Lagrange residue terms is more focused on describing the properties of complex functions as a whole, and can be quantitatively calculated. By using Taylor expansion formula with Lagrange residue, global approximation can be achieved, so that inequalities and their bounds about complex or abstract functions can be proved. If function $f(x)$ at any area of $\cup(x_0)$ of x_0 have a derivative of $(n+1)$, then for any $x\in\cup(x_0)$, so

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\S)}{(n+1)!}(x-x_0)^{n+1} \tag{4}$$

$\S$ is a value between x_0 and x.

There is also a Taylor expansion formula with Cauchy residue. The power series expansions of some elementary functions can be verified by using Taylor's formula with residual terms. For example, one can prove the identity

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n},-1<x\leq 1. \tag{5}$$

Because $f^n(x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$, so $f^n(0)=(-1)^{n-1}(n-1)!$. When $0\leq x\leq 1$, one can use the integral expression of the remaining terms [7]

$$|R_n(x)|=\left|\frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n dt\right|=\frac{x^n}{n+1}. \tag{6}$$

When $-1<x<0$, use the Cauchy residue to get that

$$R_n(x)=\frac{f^{n+1}(c)}{n!}(x-c)^n(x-a)=\frac{f^{n+1}[a+\theta(x-a)]}{n!}(1-\theta)^n(x-a)^{n+1}. \tag{7}$$

Therefore,

$$|R_n(x)|=\left|(-1)^n\frac{1}{(1+\theta x)^{n+1}}(1-\theta)^n x^{n+1}\right|\leq\frac{|x|^{n+1}}{1+\theta x}\leq\frac{|x|^{n+1}}{1+x}\to 0 \tag{8}$$

as $n\to\infty$. So, the result is exactly the Eq. (5).

3. Applications of Taylor expansion to calculate function

3.1. Finding the Limit Infinitives

Find the limit [8]

$$L=\lim_{x\to\infty^+}\sqrt[3]{x^3+3x^2}-\sqrt[4]{x^4-2x^3}. \tag{9}$$

This question can be solved by letting $\mathrm{x}=\frac{1}{\mathrm{t}}$, because $x\to\infty^+$ is not tend to 0, and put into Eq. (9) , then it is found that

$$\lim_{t\to 0}\sqrt[3]{\frac{1}{t^3}+\frac{3}{t^2}}-\sqrt[4]{\frac{1}{t^4}-\frac{2}{t^3}}=\lim_{\mathrm{t}\to 0}\frac{\sqrt[3]{1+3\mathrm{t}}-\sqrt[4]{1-2\mathrm{t}}}{\mathrm{t}}=\lim_{\mathrm{t}\to 0}\frac{(1+3\mathrm{t})^{\frac{1}{3}}-(1-2\mathrm{t})^{\frac{1}{4}}}{\mathrm{t}}. \tag{10}$$

This step is organized into a relatively familiar form, like

$$(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+\cdots. \tag{11}$$

Because the denominator is raised to the first power, the numerator expands to the first term. Namely, $(1+3t)^{\frac{1}{3}}=1+\frac{1}{3}\cdot 3t+o(t)$ and $(1-2t)^{\frac{1}{4}}=1+\frac{1}{4}\cdot(-2t)+o(t)$. So,

$$\mathrm{I}=\lim_{\mathrm{t}\to 0}\frac{(1+3\mathrm{t})^{\frac{1}{3}}-(1-2\mathrm{t})^{\frac{1}{4}}}{\mathrm{t}}=\lim_{\mathrm{t}\to 0}\frac{\frac{3}{2}\mathrm{t}+\mathrm{o}(\mathrm{t})}{\mathrm{t}}=\frac{3}{2}. \tag{12}$$

Notice that the order of expansion using Taylor’s formula is that the numerator and denominator have the smallest degree n that is not zero (cancelable), so that the terms below order n cancel out, and the limit value of the terms above order n is zero. The appearance of higher order infinitesimal does not affect the limit calculation, and the term limit of higher order infinitesimal is zero.

3.2. Prove the Equality or Inequality

Prove the inequality [9]

$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}. \tag{13}$$

Assume that $f(x) = \ln(1+x), (-1 < x < 1)$, then $f(x)$ at $x = 0$ has a Lagrange residue third order of Taylor formula $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(1+\S)^4}$ $(-1 < \S < 1)$. Because $-\frac{x^4}{4(1+\S)^4} \leq 0$, so the relation in Eq. (13) is true.

It can be seen from the above proof that the inequality is proved by Taylor's formula. First, the function is constructed, the appropriate point x is selected to expand at x_0, and then the positive and negative of the remaining terms $R_n(x)$ are judged to prove the inequality. For the problem of proving that the inequality contains the combination of elementary function, trigonometric function, transcendental function and power function, one should make full use of the Maclaurin expansion of Taylor formula at x=0, select the appropriate basic function, Maclaurin's expansion, analyze the problem, draw materials, construct and use.

3.3. Prove the Mean Value Formula

Assume $f(x)$ has a second order derivative at $[a,b]$, so prove:

$$\int_a^b f(x)\,dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{1}{24}f''(c)(b-a)^3. \tag{14}$$

For function of $F(x) = \int_a^t f(t)dt$, set $h = \frac{b-a}{2}$ and use Taylor expansion formula at $x_0 = \frac{a+b}{2}$, it is found that

$$F(x_0+h) = F(x_0) + f(x_0)h + \frac{1}{2}f'(x_0)h^2 + \frac{1}{6}f''(\S)h^3 \tag{15}$$

and

$$F(x_0-h) = F(x_0) - f(x_0)h + \frac{1}{2}f'(x_0)h^2 - \frac{1}{6}f''(\eta)h^3, \tag{16}$$

with $\S$, $\eta \in (a,b)$. Therefore,

$$\int_a^b f(x)\,dx = F(x_0+h) - F(x_0-h) = (b-a)f(x_0) + \frac{(b-a)^3}{48}(f''(\S) + f''(\eta)). \tag{17}$$

So, $f''(c) = \frac{f''(\S)+f''(\eta)}{2}$, and the relation in Eq. (14) is proved.

3.4. Prove the Power Series Expansions

Proof the expainsion [10]

$$\ln(1+x) = \sum_{n=1}^{\infty^+} \frac{(-1)^{n-1}x^n}{n}. \tag{18}$$

Because $f^n(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$, so, $f^n(0) = (-1)^{n-1}(n-1)!$ when $0 \leq x \leq 1$. The integral expression of the remaining terms is

$$|R_n(x)| = \left|\frac{1}{n!}\int_0^x (-1)^n \frac{n!}{(1+t)^{n+1}}(x-t)^n dt\right| \leq \int_0^x (x-t)^n dt = \frac{x^n}{n+1} \to 0(n \to \infty). \tag{19}$$

When $-1 < x < 0$, the Cauchy residue theorem finds that

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a) = \frac{f^{(n+1)}[a+\theta(x-a)]}{n!}(1-\theta)^n(x-a)^{n+1}. \quad (20)$$

So, $|R_n(x)| = \left|(-1)^n \frac{1}{(1+\theta x)^{n+1}}(1-\theta)^n x^{n+1}\right| \leq \frac{|x|^{n+1}}{1+\theta x} \leq \frac{|x|^{n+1}}{1+x} \to 0$ as $n \to \infty$. Thus, the relation in Eq. (18) holds.

3.5. Prove the Existence of a Condition

Assume $f(x)$, $g(x)$ can be second-order differentiability at [a,b], and $f(a) = f(b) = g(a) = 0$. For $\exists\xi \in (a,b)$, let

$$f''(\xi)g(\xi) + 2f'(\xi)g'(\xi) + f(\xi)g''(\xi) = 0 \quad (21)$$

The left-hand side of this equation can be viewed as second-order differentiability of $f(x)g(x)$ at point ξ, so prove $\exists\xi \in (a,b)$, let $[f(x)g(x)]''(\xi) = 0$ and introduce $F(x) = f(x)g(x)$. Using the Taylor expansion formula at $x = a$, it is found that

$$F(x) = F(a) + F'(a)(x-a) + \frac{1}{2}F''(\xi)(x-a)^2(a < \xi < x). \quad (22)$$

Putting $x = b$ into (22), then

$$F(b) = F(a) + F'(a)(b-a) + \frac{1}{2}F''(\xi)(b-a)^2(a < \xi < b) \quad (23)$$

Because $f(a) = f(b) = g(a) = 0$, then $F(a) = F(b) = 0$, and put $F'(a) = 0$ into (23), then $F''(\xi) = 0$, so the result is Eq. (21).

3.6. Approximation of Function

The aim of *this* section is to calculate the value of ln1.2, make the error not more than 0.0001. Let $f(x) = \ln(1+x)$, so $f(x) = \ln(1+x)$ at original of Taylor Expansion with Lagrange residue:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n x^{n+1}}{(n+1)(1+\xi)^{n+1}}(0 < \xi < 0.2). \quad (24)$$

Then

$$\ln 1.2 = ln(1+0.2) = 0.2 - \frac{0.2^2}{2} + \cdots \frac{(-1)^{n-1}0.2^n}{n} + \frac{(-1)^n 0.2^{n+1}}{(n+1)(1+\xi)^{n+1}}(0 < \xi < 0.2) \quad (25)$$

Let

$$\left|\frac{(-1)^n 0.2^{n+1}}{(n+1)(1+\xi)^{n+1}}\right| = \frac{0.2^{n+1}}{(n+1)(1+\xi)^{n+1}} < 0.2^{n+1} \leq 0.0001(0 < \xi < 0.2) \quad (26)$$

If $n = 5$, then it is found that

$$\ln 1.2 \approx 0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} - \frac{0.2^4}{4} + \frac{0.2^5}{5} = 0.2 - 0.02 + 0.00267 - 0.00040 = 0.1823. \quad (27)$$

4. Conclusion

Taylor formula is an important part of calculus. Since the content is abstract, its main idea is to use approximation method to find the value of the function. In order to solve the problem that Taylor formula is difficult to understand, the author first explains the approximate value calculation formula of the first order derivative, and finds that there are problems in the accuracy of the evaluation. This leads to the Taylor formula and its different kinds of residue form. Next, the paper introduces the application of Taylor expansion in approximate calculation, limit calculation, equality proof, inequality proof and other aspects. In doing so, they broaden the idea of learning Taylor's formula and deepen the understanding of the formula. This paper talks about the importance of Taylor's expansion formula in various forms. The significance and application of Taylor's expansion formula are also introduced. This paper gives three types of Taylor's expansion formula, namely Peano residue, Lagrange residue, and Cauchy residue. In addition, it talks about how are these types of Taylor formulas are developed. Further, this paper lists four practical problems, each of which corresponds to a problem type. The problems are to find infinitesimal values, prove equations or inequalities, prove median and prove power series. These examples show the advantages of Taylor formula in solving some typical problems in higher mathematics. It can be seen that the flexible use of Taylor formula can bring great flexibility and convenience to the solution of the problem.

References

[1] Wang Zhongmei, Meng Xianqing, He Guoman. Research on a case study on the application of Taylor formula. Journal of Datong University (Natural Science Edition), 2022, 38(01): 28-29+44
[2] Qiu Weidun. Research on the Taylor formula and its application. Journal of Longyan University, 2005, (06): 92-94.
[3] Lu Huayong. Research on the application of Taylor formula in solving problems of higher mathematics. Journal of Mathematical, physical and chemical solution research, 2021, 18: 24-26.
[4] Chen Meng, Kong Linghua. Research on the quotation of Taylor's formula. Journal of scientific advice, 2023, 21(03): 42-44.
[5] Yang xiong. Research on Taylor formula proof and application exploration. Journal of Wei fang college, 2022, 18(05): 10-15.
[6] Ma Qiaoling, Ma chao, Liu Tong. Research on the application of Taylor's formula in higher mathematics. Journal of Higher mathematics research, 2022, 25(05): 25-27.
[7] Wang Yaoge, Cui Guozhong, Guo Congzhou. Research on the application of Taylor's formula is discussed from the remaining terms of Taylor's formula. Journal of Information Engineering College, 2021, 22(05): 632-634.
[8] Qin Zhongmei. Research on a simple proof of Taylor's formula of two different residue. Journal of Higher mathematics research, 2021, 24(05): 65-67.
[9] Cao Xuexia. Research on the action of Taylor's formula on definite integrals. Journal of Scientific Advice, 2021, 43(09):63-64.
[10] Sha Ping. Research on the deductive method of Taylor Formula with Peano residual term -- a teaching practice based on exploratory teaching method. Journal of Mathematics study and research, 2021, 27(05): 152-153.