The Residue Theorem Approach to Compute Definite Integrals

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Abstract. Complex analysis is a branch of mathematics that studies the properties and behaviors of functions of complex variables, where a complex variable is a quantity that has both a real part and an imaginary part. Complex analysis is important in many areas of science, including physics, engineering, and computer science. The importance of complex analysis lies in its ability to solve problems that are difficult or impossible to solve using only real variables alone. For example, because of the complicated integrals involved, many problems in fluid mechanics, electromagnetism, and quantum mechanics can be solved using complex analysis. This paper introduces an important theorem in complex analysis, which is the residue theorem. By applying the residual theorem, the trigonometric function integral and the polynomial integral exponential function integral are calculated, which simplifies the complexity and difficulty. With the help of examples, the application of the residue theorem is demonstrated. This paper contributes to extending the idea of integral calculation and facilitates the efficient solution of integral calculations in practical problems.

Keywords: Improper integrals; Cauchy’s residue theorem; Definite integrals; Laurent series.

1. Introduction

In 1825, Cauchy’s paper about the limit of integration is a definite integral of an imaginary number stated the case of calculating real integrals. He dealt with the complex integral problem and the definition of residue. Regarding the definition, it is stated that if function $f(z)$ is holomorphic at $D(a,r)/a, r>0$, then a is isolated singularity of $f(z)$, and the residue of $f(z)$ at point a is [1]

$$\mathrm{Res}(f,a)=\frac{1}{2\pi i}\int_{|z-a|=\rho} f(z)dz, (0<\rho<r)\,. \tag{1}$$

This definition given by Cauchy is still using today, and it spread to many other subjects like differential equation and series theory. Residue theorem has become an important concept in those subjects and made a huge influence to those subjects [2].

The definition of Cauchy residue theorem is the following. Suppose U is a simply connected open subset on a complex plane, and $a_1, a_2, \ldots\ldots, a_n$, is finite points on complex plane, $f(z)$ is holomorphic function defined in $U|\{a_1, a_2, \ldots, a_n\}$. If y is a rectifiable curve that surrounds $a_1, a_2, \ldots, a_n$, but not pass any a_k, and its starting point coincides with its ending point, then [3]

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} I(\gamma, a_k)\mathrm{Res}(f, a_k)\,. \tag{2}$$

If γ is Jordan curve, then $|(\gamma, a_k) = 1$, hence,

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, a_k) \tag{3}$$

In complex analysis, the residue theorem is a powerful tool for calculating the path integral of an analytic function along a closed curve, and it can also be used to compute the integral of a real function. In this essay the author first states the method to solve integrals by using residue theorem. Three types of integrals will be introduced, together with some examples with clear process and some explanation to the processes. At the end, the author will organize all the key point and give out a summary to the paper.

2. Basis Theorems

Theorem 1 (Residue theorem). If f is holomorphic at point a, then for any extendable closed curve γ in the domain a, there is $\oint_{\gamma} f(z)dz = 0$. If a is isolated singularity of f, then the integral will not always be zero, and the integral value only depend on f and a, but not γ. Suppose the expansion equation of f at neighborhood of a is $f(z)=\sum_{n=-\infty}^{\infty} C_n (z-a)^n$, with the coefficient [4]

$$C_n = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta, n = 0, \pm 1, \pm 2, \dots. \tag{4}$$

Specially, when $n = -1$, there is $C_{-1} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)d\zeta$.

Theorem 2 (Fundamental theorem). Suppose D is a bounded domain on complex plane, and its boundary γ consists of one or more simple closed curves, if f removes its isolated singularity $z_1, z_2, \dots, z_n$ in D is holomorphic, and it is continuous without $z_1, z_2, \dots, z_n$ in domain D, then [5]

$$\oint_{\gamma} f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f; z_k]. \tag{5}$$

If f in complex field is holomorphic without $z_1, z_2, \dots, z_n$, then f has sum of residue equal to 0 at $z_1, z_2, \dots, z_n$ and $z = \infty$.

3. Applications on Definite Integrals

3.1. Trigonometric Function-type Integral

The first integral in this type is [6]

$$I = \int_0^{\pi} \frac{\cos n\theta - \cos n\theta}{\cos x - \cos\theta} dz = \pi \frac{\sin n\theta}{\sin\theta} (0 < \theta < \pi). \tag{6}$$

To begin with, it is found that $I = \frac{1}{2}\int_0^{2\pi} \frac{\cos nx - \cos n\theta}{\cos x - \cos\theta} dz = \frac{1}{2}\mathrm{Re}\int_0^{2\pi} \frac{e^{inx} - \cos n\theta}{\cos x - \cos\theta} dz$. Let $z = e^{ix}$, then $\cos x = \frac{1}{2}\left(z + \frac{1}{2}\right), dz = \frac{dz}{iz}$. Therefore,

$$I = \mathrm{Re}\lim_{\varepsilon\to 0^+} \frac{1}{i}\int_{c_1+c_2} \frac{z^n - \cos n\theta}{z^2 - 2z\cos\theta + 1} dz = \mathrm{Re}\lim_{\varepsilon\to 0^+} \frac{1}{i}\int_{c_1+c_2} \frac{z^n - \cos n\theta}{(z - e^{i\theta})(z - e^{i\theta})} dz. \tag{7}$$

From the small circle lemma, it is found that [7]

$$\lim_{\varepsilon\to 0^+}\int_{c_{\varepsilon 1}} \frac{z^n - \cos n\theta}{(z - e^{i\theta})(z - e^{-i\theta})} dz = -\pi i \lim_{z\to e^{i\theta}} \frac{z^n - \cos n\theta}{z - e^{-i\theta}} = -\pi i \frac{\sin n\theta}{2\sin\theta} \tag{8}$$

and

$$\lim_{\varepsilon\to 0^+}\int_{c_{\varepsilon 2}} \frac{z^n - \cos n\theta}{(z - e^{i\theta})(z - e^{-i\theta})} dz = -\pi i \lim_{z\to e^{-i\theta}} \frac{z^n - \cos n\theta}{z - e^{i\theta}} = -\pi i \frac{\sin n\theta}{2\sin\theta}. \tag{9}$$

By the residue theorem, it is $\int_{c_1+c_2}\frac{z^n-\cos n\theta}{(z-e^{i\theta})(z-e^{-i\theta})}dz+\int_{c_1+c_2}\frac{z^n-\cos n\theta}{(z-e^{i\theta})(z-e^{-i\theta})}dz=0$. Let $\varepsilon\to 0^+$, it is obtained that

$$\lim_{\varepsilon\to 0^+}\int_{c_1+c_2}\frac{z^n-\cos n\theta}{(z-e^{i\theta})(z-e^{-i\theta})}dz=\pi i\frac{\sin n\theta}{\sin\theta}. \tag{10}$$

Finally, it is calculated that

$$I=\mathrm{Re}\frac{1}{i}\lim_{\varepsilon\to 0^+}\int_{c_1+c_2}\frac{z^n-\cos n\theta}{(z-e^{i\theta})(z-e^{-i\theta})}dz=\pi\frac{\sin n\theta}{\sin\theta}. \tag{11}$$

The second integral is [8]

$$\int_0^{2\pi}\cos^{\ m}x\cos nx\,dx=\begin{cases}\frac{\pi}{2^{m-1}}\binom{m}{\frac{m-n}{2}},\\ 0\end{cases} \tag{12}$$

where $-m\le n\le m$ and m is even.

Consider $f(z)=\left(z+\frac{1}{z}\right)^m$. The Laurent series at z=0 is $f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$, in which the coefficient $a_n=\frac{1}{2\pi i}\oint_c\frac{f(z)}{z^{n+1}}dz$. C is the forward closed path around $x=0$. Namely,

$$a_n=\frac{1}{2\pi i}\oint_c\frac{\left(z+\frac{1}{z}\right)^m}{z^{n+1}}dz=\frac{1}{2\pi}\oint_0^{2\pi}\frac{\left(e^{ix}+\frac{1}{e^{ix}}\right)^m}{e^{i(n+1)x}}e^{ix}dx=\frac{2^{m-1}}{\pi}\int_0^{2\pi}\cos^m x\cos nx\,dx. \tag{13}$$

Expansion of the $f(z)$ in the binomial form, it is observed that [9]

$$f(z)=\sum_{n=0}^{m}\binom{m}{n}z^{m-n}\frac{1}{z^n}=\sum_{n=0}^{m}\binom{m}{n}z^{m-2n}=\sum_{\substack{n=-m\\ m-n\text{ even}}}^{m}\binom{m}{\frac{m-n}{2}}z^n. \tag{14}$$

Compare the two coefficients, it is found that when $-m\le n\le m$ and $m-n$ is even, then

$$\int_0^{2\pi}\cos^m x\cos nx\,dx=\frac{\pi}{2^{m-1}}\binom{m}{\frac{m-n}{2}}. \tag{15}$$

Otherwise, $\int_0^{2\pi}\cos^m x\cos nx\,dx=0$.

3.2. $\int_0^{+\infty}f(x)dx$-type integral

The first integral belongs to this type is [10]

$$I=\int_0^{+\infty}\frac{x^b}{1+x^2}\cos\left(ax-\frac{b}{2}\pi\right)dx \tag{16}$$

where $a\ge 0,-1<b<1$. Let $f(z)=\frac{e^{iaz}z^b}{1+z^2}$ and $0\le arg\,z\le 2\pi$, Therefore, it is found that $\mathrm{Res}[f(z);i]=\lim_{z\to e^{\frac{\pi}{2}}}\frac{e^{iaz}z^b}{z+i}=\frac{e^{-a}}{2i}e^{\frac{b\pi}{2}i}$ and $\mathrm{Res}[f(z);-i]=\lim_{z\to e^{\frac{3\pi}{2}}}\frac{e^{iaz}z^b}{z-i}=-\frac{e^{a}}{2i}e^{\frac{3b\pi}{2}i}$. Using the path integral, it is found that

$$\int_0^{+\infty}\frac{e^{iax}x^b}{1+x^2}dx=\frac{2\pi i}{1-e^{2b\pi i}}\mathrm{Res}[f(z);i]+\mathrm{Res}[f(z);-i]=-\frac{\pi}{2}\left[-\frac{cosh(a)}{cos\frac{b\pi}{2}}+i\frac{sinh(a)}{sin\frac{b\pi}{2}}\right] \tag{17}$$

Hence,

$$I = cos\frac{b\pi}{2}\int_0^{+\infty}\frac{x^b\, cos\, a\, x}{1+x^2}dx + sin\frac{b\pi}{2}\int_0^{+\infty}\frac{x^b\, sin\, a\, x}{1+x^2}dx = \frac{\pi}{2}e^{-a}. \quad (18)$$

The second integral that belongs to this type is [11]

$$I_1 = \int_{-1}^{1}(\frac{1+x}{1-x})^{m-1}\frac{dx}{x^2+1} = \int_0^{+\infty}\frac{x^{m-1}}{x^2+1}dx \quad (19)$$

and

$$I_2 = \int_{-1}^{1}\frac{\sqrt{1-x^2}}{1+x^2}dx = 2\int_0^{+\infty}\frac{x^{\frac{1}{2}}}{(x+1)(x^2+1)}dx \quad (20)$$

If $0 \le \arg z \le 2\pi$, then it is found that

$$I_1 = \int_0^{+\infty}\frac{x^{m-1}}{x^2+1}dx = \frac{2\pi i}{1-e^{2m\pi i}}\mathrm{Res}\left[\frac{z^{m-1}}{z^2+1}; i\right] + \mathrm{Res}\left[\frac{z^{m-1}}{z^2+1}; -i\right] = \frac{\pi}{2\, sin\frac{m\pi}{2}}, \quad (21)$$

and

$$I_2 = 2\int_0^{+\infty}\frac{x^{\frac{1}{2}}}{(x+1)(x^2+1)}dx = \frac{4\pi i}{1-e^{\pi i}}\sum_{k=1}^{3}\mathrm{Res}\left[\frac{z^{\frac{1}{2}}}{(z+1)(z^2+1)}; z_k\right] = \left(\sqrt{2}-1\right)\pi. \quad (22)$$

3.3. $\int_a^b f(x)dx$-type integral

The representative integral is [12]

$$I = \int_0^1\frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}dx. \quad (23)$$

By using the path integral, if upper of the secant is $argz = 0$, $arg(1-z) = 0$, then the upper integral is $\int_\delta^{1-\delta}\frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}dx$ and the lower integral is $-i\int_\delta^{1-\delta}\frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}dx$. Because of $\lim\limits_{z\to\infty} z f(z) = 0$, so $\lim\limits_{R\to\infty}\int_{C_R} f(z)dz = 0$. Then, $\lim\limits_{\delta\to 0}\int_{\gamma_0} f(z)dz = -2\pi i\lim\limits_{z\to 0} z f(z) = 0$ and $\lim\limits_{\delta\to 0}\int_{\gamma_1} f(z)dz = -2\pi i\lim\limits_{z\to 1}(z-1)f(z) = 0$. According to the residue theorem [13],

$$(1-i)\int_\delta^{1-\delta}\frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}dx + \int_{C_R+\gamma_0+\gamma_1} f(z)dz = 2\pi i\mathrm{Res}[f(z); -1] \quad (24)$$

in which

$$\mathrm{Res}[f(z); -1] = \frac{1}{2}\lim_{z\to e^{\pi i}}\frac{d^2}{dz^2}\sqrt[4]{z(1-z)^3} = -\frac{3}{64}2^{-\frac{1}{4}}e^{\frac{\pi}{4}i}. \quad (25)$$

Let $R\to\infty, \delta\to 0$, it is clear to find that

$$\int_0^1\frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}dx = \frac{2\pi i}{1-i}\cdot\left(-\frac{3}{64}2^{-\frac{1}{4}}e^{\frac{\pi}{4}i}\right) = \frac{3\sqrt[4]{2}}{64}\pi. \quad (26)$$

4. Conclusion

Residues play an essential role in the field of functions of complex variables. In this paper, the problem of residues in the field of complex analysis has been discussed. This paper gives the basic knowledge and related concepts related to the residue theorem, and obtains the definition of the Cauchy residue theorem. Further, the paper gives an application of the theory to several. In this essay, the author states several methods to solve integrals by using residue theorem, and there are three types of integrals for each type. The residue theorem is a fundamental concept in complex variable function theory that provides a powerful tool for solving a wide range of improper integral problems involving real variable integrands. The definite integral is calculated by using the residue number theorem. With its help, integrals can be calculated in a clear and intuitive manner, the residue theorem offers a valuable approach for solving complex problems that may be challenging to approach through other methods. In conclusion, with the help of example problems, this paper is helpful for the method and idea of improper integral calculation and promote the efficient solution of integral calculation in practical problems.

Authors Contribution

All the authors contributed equally and their names were listed in alphabetical order.

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