Calculating Integrals Pertaining Trigonometric Function and Polynomial Function by Residue Theorem

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Abstract. Within the domain of complex analysis, the residue theorem, occasionally referred to as Cauchy's residue theorem, emerges as a formidable and indispensable technique for the assessment of line integrals along closed curves involving analytic functions. Remarkably, this unique trait makes it a prized tool in the mathematician's arsenal, which frequently finds application in the calculation of both real integrals and infinite series. Cauchy's Residue Theorem simplifies the solution of complex integrals by focusing on singularity points, offering a streamlined and efficient alternative approach to the complexities associated with conventional integral methods. This paper utilizes trigonometric and polynomial functions as illustrative examples to underscore the significance of the residue theorem in the assessment of improper integrals. Furthermore, it offers in-depth examinations of three distinct examples for each category. These comprehensive evaluations serve to emphasize the importance of the residue theorem in the field of complex analysis and highlight its versatility in addressing a wide range of mathematical challenges.

Keywords: Calculus; Residue theorem; Trigonometric function; Polynomial function.

1. Introduction

The residue theorem, often known as Cauchy's residue theorem, is a significant theorem in complex analysis and an invaluable technique for evaluating line integrals of analytic functions over closed curves and computing real integrals and infinite series. It generalizes the Cauchy integral theorem and the Cauchy integral formula. In brief, the residue theorem states that the integral of function $f(z)$ around a closed contour enclosing a single pole of $f(z)$ is $2\pi i$ times the residue at the pole [1]. More specifically, if a function is holomorphic, in other words, complex differentiable throughout the complex plane except at a limited number of isolated singularities, then the value of a specific type of integral can be calculated through the addition of the function's residues at each of these singularities [2]. Cauchy is credited for developing and establishing the Residue Theorem and has been referred to as "a revolutionary in mathematics and a highly original founder of modern complex function theory" [3]. The contributions of earlier mathematicians in fields like complex variables, complex function theory, and differential equations allowed Cauchy to accomplish the greatest fundamental theorem in complex analysis and contribute far more than anybody else.

The generalized integral is the simplest and most practical application of the residue theorem, while there are several directions in which it can be applied. As an illustration, the research paper examines the application methods and skills of the generalized residue theorem in the Laplace inverse transform [4]. The residue theorem has applications in algebraic geometry, Abelian integrals, functional analysis, linear algebra, quantum field theory, and dynamical systems [5]. The theorem has numerous real-world applications in addition to solving various real integral and complex integral types. Inspired by this theorem, multiple researchers have investigated and attempted to use this theorem in various academic fields including mathematics, engineering, and physics. Existing literature discusses quantum theory of one-dimensional free electron gas by means of residue theorem and exponential excitation response of electric network circuits using this approach.

This paper will introduce the applications of the Residue Theorem in complex planes, which evaluates two different types of improper integrals involving multiple valued functions and trigonometric functions. Each section will assess three concrete examples in details and explain each step of the analysis. In addition, the essay not only defines the Cauchy Residue Theorem and discusses the proof of the theorem, but also explains the development of the theorem and its origins.

2. Residue Theorem

In complex analysis, through the study of the relationship between the coefficient of the negative power term of the Laurent series and the curve integral of the function, the concept of the Residue theorem is produced [6]. The most basic theorem, the Cauchy's residue theorem, is developed by Cauchy's integral formula [7]

$$f(z_0) = \frac{1}{2\pi i}\pi i \oint \frac{f(z)}{z - z_0} dz\,. \tag{1}$$

It is a powerful tool for calculating the path integral of analytic functions along closed curves.

Next, the paper shall prove this theorem. Let R has a piecewise smooth boundary and is bounded. With the exception of a finite number of isolated singularities $z_1, z_2, \cdots z_n$, it is assumed that f is holomorphic on $R \cup \partial R$. Then [8]

$$\int_{\partial R} f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)\,. \tag{2}$$

Let D_i be the disc with a sufficiently small radius that is centered at *zi*, then

$$\int_{\partial D_i} f(z)dz = 2\pi i \cdot \mathrm{Res}(f, z_i). \tag{3}$$

R_0 can be subdivided into finite simply connected closed domains if one considers R_0 to be the domain of R minus D_i. As f is holomorphic on $R \cup \partial R$, therefore, on those simply connected closed fields and their boundaries. The above comment shows that the integral along the boundary is equal to zero, which can get $\int_{\partial R_0} f(z)dz = 0$. [8]. Observe that

$$\int_{\partial R_0} f(z)dz + \sum_{i=1}^{n} \int_{\partial D_i} f(z)dz = \int_{\partial R} f(z)dz\,, \tag{4}$$

then it is easy to find that [8]

$$\int_{\partial R} f(z)dz = \sum_{i=1}^{n} \int_{\partial D_i} f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i). \tag{5}$$

In the process, the knowledge of Laurent series and zero and poles is used. The Laurent series is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n\,, \tag{6}$$

in which $c \in \mathbb{C}$. Laurent's series extends Taylor's series to a negative power. At the point $z \neq c$, $f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ (normal part) $+ \sum_{n=1}^{\infty} a_{-n} \left(\frac{1}{z-c}\right)^n$ (principal part) [9]. The root of the denominator polynomial of the transfer function is called the pole of the system, and the root of the numerator polynomial is called the zero of the system. That's the definition of pole and zero [10].

3. Applications

3.1. $R(\cos\theta, \sin\theta)$-type Integral

The first integral of this class is

$$I = \int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta}. \tag{7}$$

The initial step is to let $z = e^{i\theta}$ and introduce a complex variable based on $z = e^{i\theta}$. By using the Euler's formula, $\cos\theta = \frac{z}{2} + \frac{1}{2z}$ and $d\theta = \frac{dz}{iz}$. Therefore, the Eq. (7) can be rewritten as

$$I = \int_{C:|z|=1} \frac{\frac{dz}{iz}}{1+\left(\frac{z}{2}+\frac{1}{2z}\right)^2} = \int_{C:|z|=1} \frac{4zdz}{i(z^4+6z^2+1)}. \tag{8}$$

θ is in the interval of 0 to 2π, so z orbits around a closed contour C which is a unit circle $|z| = 1$ for one cycle. Suppose $z^2 = u$, then u orbits around this contour for two cycle and I becomes the integral of u, namely

$$I = 2\int_C \frac{2du}{i(u^2+6u+1)} = \frac{4}{i}\int_C \frac{du}{u^2+6u+1}. \tag{9}$$

The integrand only has one singular point is located inside the contour that is $u = \sqrt{8} - 3$. Therefore, the residue is $\underset{u=-3+\sqrt{8}}{\mathrm{Res}} f(u) = \frac{1}{u+3+\sqrt{8}}\Big|_{u=-3+\sqrt{8}} = \frac{1}{2\sqrt{8}} = \frac{1}{4\sqrt{2}}$. As a result, by using the residue theorem, it can be calculated that

$$I = \frac{4}{i}\cdot 2\pi i \cdot \frac{1}{4\sqrt{2}} = \sqrt{2}\pi. \tag{10}$$

In the same way, the same Residue theorem can also be applied to solve another integral

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta, \tag{11}$$

where $a > b > 0$ and a, b are constants.

To start with, let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta = izd\theta$, $d\theta = \frac{dz}{iz}$ as before. Then $\sin\theta = \frac{1}{2i}(z - z^{-1})$, $\cos\theta = \frac{1}{2}(z + z^{-1})$. So, the above integral equals to

$$I = \int_{|z|=1} -\frac{(z-z^{-1})^2}{4}\cdot\frac{1}{a+b\left(\frac{z+z^{-1}}{2}\right)}\cdot\frac{dz}{iz}. \tag{12}$$

In the fraction $-(z-z^{-1})^2/4$, both the numerator and denominator are multiplied by z^2, and in the fraction $1/a + b\left(\frac{z+z^{-1}}{2}\right)$, they are multiplied by z simultaneously. Hence, the integral in Eq. (12) becomes

$$I = \int_{|z|=1}\left[\frac{-(z^2-1)^2}{4z^2}\right]\cdot\frac{1}{a+b\left(\frac{z^2+1}{2z}\right)}\cdot\frac{dz}{iz} = \frac{i}{2b}\int_{|z|=1}\frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}dz, \tag{13}$$

in which $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$. Because $a > b > 0$, so it is obvious that it needs to compare α and β with $\frac{b}{b}$. If α, β are less than 1, then they are inside the contour. On the contrary, they are not inside it. Obviously, α lies into the unit circle and β is outside. Therefore, inside the circle $|z| = 1$, it only has one second-order pole at $z = 0$ and one first-order pole at $z = \alpha$. So, at

$z=0$, it is $\mathrm{Res}(f(z), z=0) = \left[\frac{(z^2-1)^2}{z^2+\frac{2a}{b}z+1}\right]'\Big|_{z=0} = -\frac{2a}{b}$, and at $z=\alpha$, it is $\mathrm{Res}(f(z), z=\alpha) = \frac{(z^2-1)^2}{z^2(z-\beta)}\Big|_{z=\alpha} = \frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)} = \frac{2\sqrt{a^2-b^2}}{b}$. Thus, the original integral turn into

$$I = \frac{2\pi}{b^2}\left[a - \sqrt{a^2-b^2}\right]. \tag{14}$$

The third integral of this type is

$$I = \int_0^{\pi} \frac{\cos mx}{5-4\cos x}\,dx. \tag{15}$$

Because the integrand is an even function with respect to x, so the original function can be written as $I = \frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos mx}{5-4\cos x}dx$. Let $\mathrm{I}_1 = \frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos mx}{5-4\cos x}dx$, and $I_2 = \frac{1}{2}\int_{-\pi}^{\pi}\frac{\sin mx}{5-4\cos x}dx$, therefore

$$I_1 + iI_2 = \int_{-\pi}^{\pi}\frac{e^{imx}}{5-4\cos x}dx. \tag{16}$$

Then, suppose $z = e^{ix}, dz = ie^{ix}\,dx,\ dx = \frac{dz}{iz}$ as before, the Eq. (16) equals to

$$\frac{1}{i}\int_C \frac{z^m}{5z-2(1+z^2)}dz = \frac{i}{2}\int_C \frac{z^m}{\left(z-\frac{1}{2}\right)(z-2)}dz. \tag{17}$$

Inside the unit circle *C*, the integrand has only one pole of first order that is $\mathrm{z} = \frac{1}{2}$, with $\mathrm{Res}_{z=\frac{1}{2}} f(z) = \frac{z^m}{z-2}\Big|_{z=\frac{1}{2}} = -\frac{1}{3\cdot 2^{m-1}}$. Because of the Residue theorem, it is found that

$$I_1 + iI_2 = -\frac{1}{2i}\cdot 2\pi i\cdot\left(-\frac{1}{3\cdot 2^{m-1}}\right) = \frac{\pi}{3\cdot 2^{m-1}}. \tag{18}$$

Hence, $I_1 = \frac{\pi}{3\cdot 2^{m-1}}$, $I_2 = 0$, and the original function can be calculated as

$$I = \frac{1}{2}I_1 = \frac{\pi}{3\cdot 2^m}. \tag{19}$$

3.2. $\frac{\mathbf{P(x)}}{\mathbf{Q(x)}}$ - type Integral

The first example of this class is

$$I = \int_0^{\infty}\frac{1}{(x^2+1)^2}dx \tag{20}$$

To begin with, it is useful to let $F(z) = \frac{1}{(z^2+1)^2}$. By using the residue theorem and applying it to the example, it is found that $\int_C F(z)dz = \int_{-R}^{R}F(x)dx + \int_{C_R}F(z)dz$, which equals to $2\pi i\,\mathrm{Res}_{z=i}[F(z)]$. By rearranging the equations, the above equations can be rewritten as following $2\pi i\,\mathrm{Res}_{z=i}[F(z)] = 2\int_0^R F(x)dx + \int_{C_R}F(z)dz$. Here, the integral is

$$\int_0^R F(x)dx = \pi i\left[\mathrm{Res}_{z=i}[F(z)]\right] - \frac{1}{2}\int_{C_R}\frac{dz}{(z^2+1)^2}. \tag{21}$$

Then, the residue at $z=i$ is calculated, which gives $\mathrm{Res}_{z=i}[F(z)] = \lim_{z\to i}\left[\frac{d}{dz}(z-i)^2\frac{1}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i}\left[\frac{-2}{(z+i)^3}\right] = \frac{-2}{(2i)^3} = \frac{1}{4i}$. After this, the determined residue is substituted into Eq.

(21). Hence, $\int_0^R \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} - \frac{1}{2}\int_{C_R} \frac{dz}{(z^2+1)^2}$. Since $\int_{C_R} F(z)$ vanishes as R approaches infinity, therefore $\lim_{R\to\infty}\left[\frac{\pi R}{(R^2-1)^2}\right] = 0$ and $\lim_{R\to\infty}\left[\int_0^R \frac{dx}{(x^2+1)^2}\right] = \frac{\pi}{4}$. Hence, the residue of the original integral is

$$I = \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}. \quad (22)$$

Likewise, the same residue theorem can be applied to solve the second integral

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1} dx \quad (23)$$

In order to compute this integral, it is necessary to introduce a "keyhole contour". Let C be the "keyhole" contour, with A and B along the real interval (r, R) where R stands for the outer radius and r represents the inner radius, and use the positive x-axis as the branch cut. Since $Z^{1/2} = e^{(\frac{1}{2})(\log_e|z|+iarg(z))}$, an argument of 0 along A and an argument of 2π along B are employed. The real integral wanted is part of $\lim_{r\to 0}\lim_{R\to\infty}\oint_C \frac{z^{1/2}}{z^2+1}dz$. Because the integrand has simple poles at $z = i$ and $z = -i$. So, at $z = i$, $\text{Res}\left(\frac{z^{\frac{1}{2}}}{z^2+1}, i\right) = \lim_{z\to i}(z-i)\frac{z^{\frac{1}{2}}}{(z+i)(z-i)} = \frac{i^{\frac{1}{2}}}{2i} = \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$. Similarly, at $z = -i$, $\text{Res}\left(\frac{z^{\frac{1}{2}}}{z^2+1}, -i\right) = \lim_{z\to -i}(z+i)\frac{z^{\frac{1}{2}}}{(z+i)(z-i)} = -\frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$. Thus,

$$\oint_C \frac{z^{1/2}}{z^2+1}dz = 2\pi i\left(\frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}\right) = \pi\sqrt{2}. \quad (24)$$

Noticing that the $\int_C = \int_A + \int_{C_R} + \int_B + \int_{C_r}$. As r approaches 0 and R approaches infinity, one has that $\int_A \to \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx$, $\int_{C_R} \to 0$ and $\int_{C_r} \to 0$ by using the estimation lemma-inequality. Hence, along B, $\frac{z^{\frac{1}{2}}}{z^2+1} = \frac{e^{\frac{1}{2}(\log_e|z|+i2\pi)}}{z^2+1} = \frac{\sqrt{x}e^{i\pi}}{x^2+1}$, so $\int_B \to \int_\infty^0 \frac{-\sqrt{x}}{x^2+1}dx = \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx$. Thus,

$$I = \int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi\sqrt{2}}{2} \quad (25)$$

The last example available is to compute

$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}. \quad (26)$$

The same idea is applied as above. In doing so, a closed contour and a related integrand are chosen, then the residue theorem should be applied to the contour integral. Let C be the unit circle $z(\theta) = e^{i\theta}$, with $0 \le \theta \le 2\pi$. So $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}(z - z^{-1})$ and $dz = ie^{i\theta}d\theta$, which means $d\theta = \frac{dz}{iz}$. Therefore, the real integral equals the contour integral

$$I = \oint_C \frac{dz}{iz\left(\frac{5}{4} + \frac{z - z^{-1}}{2i}\right)} = \oint_C \frac{4}{2z^2 + 5iz - 2}dz. \quad (27)$$

Since there are two simple poles at $z = -2i$ and $z = -i/2$, and contour C is a unit circle cantered at 0, only the pole $z = -i/2$ will be enclosed by the contour C, thus only this pole is required to find the residue. Therefore,

$$I = \oint_C \frac{4}{2z^2 + 5iz - 2} dz = \oint_C \frac{4}{(z + 2i)(2z + i)} dz = 2\pi i \,\mathrm{Res}\left(\frac{4}{(z + 2i)(2z + i)}, -i/2\right). \quad (28)$$

Now $\lim_{z \to -i/2} \left(z - \left(-\frac{i}{2}\right)\right) \frac{4}{(z+2i)(2z+i)} = \frac{4}{3i} = \frac{8}{3}\pi$. Hence,

$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta} = \frac{8}{3}\pi. \quad (29)$$

4. Conclusion

To sum up, this paper uses the residual theorem to solve two different definite integrals. The main idea of the residual theorem is to transfer the path integral to the sum of residuals of points in the region contained by the path. In this paper, the types of integrals considered include trigonometric function and rational expression. Under the verification of several examples, the method of solving the problem can be summarized. For trigonometric integrals, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ is the preferred entry point and then complex numbers are involved. For integrals of rational expression type, factorization is performed first and then residue is obtained. In this question type, a keyhole contour is introduced. The residue theorem makes it easier to solve integral problems. In the first half of this paper, the proof of residue theorem and its historical origin, the definition and application of singularities, and the introduction of Laurent's series and Taylor expansion are given. In addition, there are two other basic categories, which are the types of integrals of logarithmic functions and exponential functions. Obviously, the residue theorem is indeed an effective method to solve the integration of analytic functions along a path and the integration of some ordinary functions.

Authors Contribution

All the authors contributed equally and their names were listed in alphabetical order.

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