

# Integrating Complex Functions in Certain Situations by Two Methods

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**Abstract.** This article will mainly discuss several ways of integrating complex functions. In real field, the basic idea of integrating a function is to find its anti-derivative. However, it is challenging to find an anti-derivative of a complex function that is not differentiable at some points. This is because that in complex field, the derivative of a differentiable complex function must be also differentiable. Thus, finding the anti-derivative of non-differentiable function in complex plain is unapproachable. Moreover, even though a differentiable complex function is given, computing the anti-derivative will be very complicated or time-consuming. As a result, exclusive methods are required to calculate the integral of complex functions. Fortunately, Cauchy Residue Theorem and Cauchy-Goursat Theorem provide comprehensive ways for computing the integrals. It is crucial to realize their importance and they will be discussed in the following context. Additionally, the specific steps of application will also be shown. This study demonstrates clearly that the Residue theorem is a power method to handle with a series of complex integrals.

**Keywords:** Cauchy Residue Theorem; Cauchy-Goursat Theorem; complex integral.

## 1. Introduction

The history of complex number can be traced back to sixteen centuries, when Jerome Cardan wrote the square root of a negative number in the formula. After that, René Descartes mentioned the term “imaginary number” in his book. For a long time, lots of mathematicians dedicated to the study of calculus theory for complex functions. Some of them have taken a big step in this field. For instance, Euler formula and Cauchy Riemann Equation are two fundamental and important foundation of the field of complex integration. When talking about the integral of a complex function, the first question is how to integrate a complex function, which is the main topic of this work. Thus, this essay will mention some methods and special cases while integrating a function.

Apparently, many mathematicians have worked on this for many years. For instance, Su and Yu discussed some special cases while integrating a function [1]. On the other hand, many authors focus on the teaching aspect such as the course of complex function and integral transformation [2]. As shown in those work, the basic theory of integrating a function is quite comprehensive. The integrations of certain function or some special cases, however, are still topics that can be explored in a broader vision.

In the following section, the section 2 will talk about different methods for integrating different kinds of complex functions. To be more specific, two theorems will be mentioned and their proves will be provided respectively. Then, in section 3 four examples will be given, which are different types questions of how to integral a complex function. Section 4 will re-emphasize the key idea of this essay and point out the reason why this essay is written.

## 2. Different methods for integrating complex functions

### 2.1. Cauchy-Goursat Theorem

Cauchy-Goursat Theorem (also known as Cauchy Integral Theorem [3]) implies that if  $f(z)$  is analytic everywhere in a region  $R$ , then

$$\oint_{\gamma} f(z) dz = 0 \quad (1)$$

where  $\gamma$  is a Jordan curve in region  $R$  [4]. To prove this theorem, it is useful to write  $z$  as  $z \equiv x + iy$  and  $f(z)$  as  $f(z) \equiv u + iv$ , where  $u \equiv f(x, y)$  and  $v \equiv g(x, y)$ . Thus, the formula in Eq. (1) can be rewritten as

$$\int_{\gamma} (u + iv)(dx + idy) = \int_{\gamma} [f(x, y)dx - g(x, y)dy] + i \int_{\gamma} [g(x, y)dx + f(x, y)dy]. \quad (2)$$

By using Green's theorem [5], it gives that

$$\int_{\gamma} [f(x, y)dx - g(x, y)dy] = - \iint \left( \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \right) dx dy \quad (3)$$

and

$$\int_{\gamma} [f(x, y)dx + g(x, y)dy] = \iint \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy. \quad (4)$$

Joining together, the Eq. (2) becomes

$$\oint_{\gamma} f(z) dz = - \iint \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx dy + i \iint \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy. \quad (5)$$

In addition, Cauchy-Riemann equations [6] require that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (6)$$

Taking (6) into (5), it is immediately to find that

$$\oint_{\gamma} f(z) dz = 0, \quad (7)$$

and the Cauchy-Goursat Theorem is proven.

## 2.2. Residue Theorem

If function  $f$  is not differentiable at some points  $z_0, z_1, \dots, z_n$  in a certain region  $R$ , then the integral of the function can be calculated by the given formula

$$\int_C f(z) dz = 2\pi i \sum_{k=0}^n \text{Res}[z_k, f(z)] \quad (8)$$

where  $C$  is a simple closed curve completely inside region  $R$ .  $\text{Res}(f, z_n)$  means the residue of function  $f$  at point  $z_n$ . By using the residue theorem (which is also known as Cauchy Residue Theorem [7]), it is more convenient to compute the integrals of some complex functions.

*Proof.* Laurent expansion [8] tells that if a function  $f$  is analytic everywhere in an annular region  $R_1 < |z - z_0| < R_2$ , which the center of the region is at point  $z_0$ .  $C$  is a Jordan curve [4] around point  $z_0$  and also lying inside the region. Then,  $f(z)$  can be expanded to

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2). \quad (9)$$

In this case,  $n = 0, 1, 2, \dots$ , and

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}, b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}. \quad (10)$$

Notice that in equation (10), when  $n = 1$ , then the denominator of the fraction inside the integral will become 1, which becomes  $b_1 = \frac{1}{2\pi i} \int_C f(z)dz$ , and the final equation will be

$$\int_C f(z)dz = 2\pi i b_1. \quad (11)$$

In Laurent expansion,  $b_1$  is the coefficient of  $\frac{1}{z - z_0}$ , which is called the residue of the function at point  $z_0$  [8]. Thus,  $b_1$  can also be written as  $b_1 = \text{Res}[z_0, f(z)]$  and Eq. (11) will be

$$\int_C f(z)dz = 2\pi i \text{Res}[z_0, f(z)]. \quad (12)$$

As a result, if there are several points  $z_0, z_1 \dots z_n$  that meet the requirements of the Laurent expansion, the Eq. (12) can be generalized to

$$\int_C f(z)dz = 2\pi i \{ \text{Res}[z_0, f(z)] + \text{Res}[z_1, f(z)] + \dots + \text{Res}[z_n, f(z)] \}, \quad (13)$$

which is equivalent to Eq. (1).

### 3. Applications

In this section, some examples will be provided and the problems will be solved by methods mentioned in the previous section.

#### 3.1. Example 1

The first example is to find out  $I_1$ , which is [9]

$$I_1 = \int_0^{2\pi} (z^3 + 2z^2 - 9)dz. \quad (14)$$

Notice that the function is analytic everywhere in the region. According to the Cauchy-Goursat theorem, the value of the integral is zero, which is

$$I_1 = \int_0^{2\pi} (z^3 + 2z^2 - 9)dz = 0. \quad (15)$$

#### 3.2. Example 2

The second example is to find out the value of  $I_2$ , which is

$$I_2 = \int_0^{2\pi} \frac{\sin 2\theta d\theta}{1 + \cos 2\theta}. \quad (16)$$

In this case, the function can be first transformed to the form of  $I_2 = \int_0^{2\pi} f(z)dz$ . According to the Euler's formula [10], it is easy to check that  $z^2 = e^{i2\theta} = \cos 2\theta + i \sin 2\theta$  and  $z^{-2} = \cos 2\theta - i \sin 2\theta$ . Thus,  $\cos 2\theta = \frac{z^2 + z^{-2}}{2} = \frac{z^4 + 1}{2z^2}$  and  $\sin 2\theta = \frac{z^2 - z^{-2}}{2i} = \frac{i(1 - z^4)}{2z^2}$ . Meanwhile, notice that  $d\theta = \frac{dz}{iz}$ , the equation will be

$$I_2 = \int_0^{2\pi} \frac{\sin 2\theta d\theta}{1 + \cos 2\theta} = \int_0^{2\pi} \frac{i(1 - z^4)}{2z^2 + z^4 + 1} \frac{dz}{iz} = \int_0^{2\pi} \frac{1 - z^4}{z[(z + i)(z - i)]^2} dz. \quad (17)$$

Notice that the undifferentiable points are  $z_0 = 0, z_1 = i, z_2 = -i$  three points, and they all lie in the contour  $C$ , which is inside a circle defined by  $|z| = 1$ . Therefore, the value of  $I_2$  will be

$$I_2 = 2\pi i \sum_{n=0}^2 \text{Res} \left[ z_n, \frac{1 - z^4}{z[(z + i)(z - i)]^2} \right]. \quad (18)$$

By calculating residue of function at point  $0, i, -i$ , respectively, the final computation will be

$$I_2 = 2\pi i(1 + i + 1) = 4\pi i - 2\pi. \quad (19)$$

In addition, if the contour changes, then the value of the  $I_2$  may change same time. For instance, if the contour contains only  $z_0$ , then  $I_2$  will be just  $2\pi i$ , which exclude the residues of the function at point  $z_1, z_2$ . Moreover, if the contour rotates around three points clockwise two times, then the value of  $I_2$  should be two times bigger to the opposite number of its original value, which in this case the final value of  $I_2$  will be  $4\pi - 8\pi i$ .

### 3.3. Example 3

This example shows another way of finding the residue of the function  $f$ , which is

$$f(z) = \frac{19z - 7}{z^2 - 1}. \quad (20)$$

Instead of directly calculating the residue of function at point  $1$  and  $-1$ , another method is to split the function into two fractions which is

$$f(z) = \frac{A}{z + 1} + \frac{B}{z - 1} \quad (A, B \in \mathbb{R}). \quad (21)$$

Notice that  $z^2 - 1$  can be split into two factors  $z + 1$  and  $z - 1$ . While doing the reduction of the fraction to a common denominator, the denominator will be  $z^2 - 1$  again. The advantage of change the fraction into such form is that two fractions in (2) will be same as the expression of the Laurent expansion. In this case,  $A$  and  $B$  is the residue of the function at point  $-1, 1$  respectively. Thus, the goal is to find the value of  $A$  and  $B$ . To find the values, two fractions can be added together

$$f(z) = \frac{A}{z + 1} + \frac{B}{z - 1} = \frac{A(z - 1) + B(z + 1)}{z^2 - 1} = \frac{(A + B)z - (A - B)}{z^2 - 1}. \quad (22)$$

Compare Eq. (22) with Eq. (20), the result will be

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 19 \\ 7 \end{bmatrix}, \quad (23)$$

and it is found that  $A = 13$  and  $B = 6$ . Therefore, the result is that  $\text{Res}[-1, f(z)] = 13$  and  $\text{Res}[1, f(z)] = 6$ . If the integral of function needs to be calculated, such as

$$I_3 = \int_C f(z) dz, \quad (24)$$

where contour  $C$  is a simple closed curve that contain both undifferentiable points, then

$$I_3 = \int_C f(z) dz = 2\pi i(A + B) = 38\pi i. \quad (25)$$

### 3.4. Example 4

Example 4 is an extension of example 3, where a more general solution to be found is when

$$f(z) = \frac{g(z)}{z^n - b}. \quad (26)$$

In this case,  $g(z)$  is a function that is differentiable everywhere and  $b$  is a non-zero constant. To find the residue of this function  $f$ , firstly notice that the points that undifferentiable are  $z_k$  such that  $z_k^n = b$ . Then, using Taylor expansion to expand function respectively. First, the expansion of  $\frac{1}{z^n - b}$  can be gotten indirectly by expanding  $z^n - b$  at point  $z_k$  first. So,

$$z^n - b = 0 + nz_k^{n-1}(z - z_k) + \frac{n(n-1)z_k^{n-2}(z - z_k)^2}{2} + \dots \quad (27)$$

Notice that since  $z_k^n = b, b \neq 0$ ,  $z_k$  is also non-zero, which divide both side by  $z_k$ , the equation will be  $z_k^{n-1} = \frac{b}{z_k}$ , take back to Eq. (27), it is found that

$$z^n - b = \frac{bn}{z_k}(z - z_k) \left[ 1 + \frac{(n-1)z_k^{n-1}(z - z_k)}{2b} + \dots \right]. \quad (28)$$

In this case, only the second term of the Taylor expansion is useful. So, the expansion above all stops at the third term. After find out the Taylor expansion of  $z^n - b$ , then find the reciprocal which is

$$\frac{1}{z^n - b} = \frac{z_k}{bn} \frac{1}{z - z_k} [1 + a_1(z - z_k) + a_2(z - z_k)^2 + \dots] \quad (29)$$

When taking the reciprocal, the part inside the middle bracket is useless. As a result,  $a_1, a_2 \dots$  are used to represent some constants in front of the  $(z - z_k)^n$  terms. After that, the expansion of  $f(z_k)$  can be calculated by computing  $g(z_k) \frac{1}{z_k^n - b}$ , which is

$$f(z_k) = \frac{z_k}{bn} \frac{1}{z - z_k} [g(z_k) + a_1g(z_k)(z - z_k) + g'(z_k)(z - z_k) + \dots]. \quad (30)$$

Since Eq. (30) can be reckoned as a Laurent series, the only constant that is useful is the constant of the term  $\frac{1}{z - z_k}$ . After further expansion of Eq. (30), the constant is  $\frac{z_k g(z_k)}{bn}$ , which by definition is the residue of function  $f$  at point  $z_k$ . Thus, a conclusion can be made that

$$Res[z_k, f(z_k)] = \frac{z_k g(z_k)}{bn}. \quad (31)$$

For instance, example 3 can be solved by recognizing that  $b = 1, n = 2, g(z) = 19z - 7$ , take  $z_k = -1$  and  $z_k = 1$  into the formula, the answer is the same. In this special case, where denominator contains only one  $z^n$  term and one constant term, a general formula for computing the integrals is

$$\int_C f(z) dz = 2\pi i \sum_{i=0}^k \frac{z_i g(z_i)}{bn} \quad (32)$$

Where  $z_1, z_2, \dots, z_k$  are points that undifferentiable in  $f(z)$ ,  $g(z)$  is the numerator of  $f(z)$ ,  $b$  is the constant term in the denominator and  $n$  is the exponent of  $z$  in the denominator.

## 4. Conclusion

This paper mainly talks about two frequently-used methods of integrating complex functions and several practical examples that clarify the detailed steps of how to use those methods on certain questions. The uniqueness of this article lies in finding out more convenient methods when the

function has some certain patterns or can be generalized to some form, which mentioned in the section three, and solve the problem quicker. The tools have already been given by Mathematicians long time ago. It is notable that the tools are universal considering the different patterns and elements that should be considered in different problems or functions. Nonetheless, when those different situations are classified, a simpler expression of the formula for each situation will be revealed. Although it is more difficult to remember all special situations than just only remember the general formula, the convenience of those special formulas cannot be ignored. Those formulas, evidently, can greatly reduce the redundancy while doing the computation. Hopefully the highlighting of each special case will improve the calculation speed and derive new theories.

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