Introduction to Integration and Techniques to Calculate Integrals

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Abstract. Integrals are the powerful tools that provide deep insights into the behavior of functions and are essential for modelling and understanding the world of human being. Integration has widespread applications across various fields. Integrals are used to calculate accumulation processes and find the area under curves. This is crucial in physics by calculating displacement from velocity and in various engineering and economic applications by calculating total profit from a production function. Integrals are used in probability theory and statistics for calculating probabilities and expected values. In statistical applications, integration plays a significant role in calculating areas under probability distributions, which are used in hypothesis testing and confidence intervals. Solving a complex integral is too tough if just use some fundamental formulas. The techniques of integration include substitutions, integral by parts, integration using Trigonometric Identities, fixed formulas of integration. This essay is going to introduce some of the most important methods of integration.

Keywords: Real analysis; Calculus; Definite integral; Integration by parts.

1. Introduction

By the time of Archimedes, 200 BC, there was already the "exhaustion method" that was very similar to the modern integral method in calculating the area of the graph surrounded by a parabola and a straight line. By the 12th century, Bhasgara of India had developed a predecessor method of integration. In the 17th century, Newton integrated the method of differentiation and integration, and tried to deduce the law of motion of celestial bodies from the theory of gravitation [1].

From the definition and geometric meaning of the definite integral, the definite integral of the function $y = f(x)$ in $[a, b]$ is equal to the algebraic the summation of the area under the curve enclosed by the functions $y = f(x)$. Thus, the upper range is b and lower range is a, and put b and a into this function respectively and use $f(b) - f(a)$ can get the answer. The area of a trapezoid with a curved edge can be found by finding the definite integration of the function. For example, the problem of maximum value can be solved by definite integral and the increasing rate also can be handled by definite integral [2]. Let the variable y be a function of time t, i.e., $y = f\ (t)$, then the ratio is the relative change of the function $f(t)$ in the time interval. If $f(t)$ is differentiable, the limit is defined as the instantaneous growth rate of the function $f(t)$ at point t in time. Making the function $y = f(x)$ be differentiable at point x, the ratio of the relative change of the function $\varDelta yy = f(x + \varDelta x) - f(x)y$ to the relative change of the independent variable $\varDelta xx$, the limit when $\varDelta x \to 0$ is called the relative change rate of the function $y = f(x)$ at point x, or called the elastic function [3].

In this paper, the section 2 concentrates on how to integrate different functions in different situations and there are all the methods to integrate like substitution, numerical integration, and integration by part. In addition, the fixed integral formulas of some special function are also indicated in the section 2. The next section is to apply the techniques which are mentioned in section 2 in the real integral problems.

2. Techniques of Integration

An integral represents the sum of discrete data. Integrals are calculated to find functions that describe area, displacement, and volume resulting from a small set of data that cannot be measured individually. In calculus, the concept of limits is used to implement algebra and geometry [4]. Limits are useful for studying the consequences of points on a graph, such as how they approach each other until their distance approaches zero. Table 1 gives some results for representative examples.

Table 1. Results of several indefinite integrals. Some constant terms are omitted.

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$\int x^{-1}\,dx = \ln\lvert x\rvert + C$	$\int e^x = e^{\mathrm{x}} + \mathrm{C}$
$\int \sin x\,dx = -\cos x$	$\int \cos x dx = \sin x + C$	$\int \tan x dx = \lvert\sec x\rvert + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \frac{1}{1+x^2} dx = arc\tan x + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = arc\sin x + C$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert$	$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert$	$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left\lvert x + \sqrt{x^2-a^2}\right\rvert$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left\lvert x + \sqrt{x^2+a^2}\right\rvert$

However, most problems are not so simple. If a given function is differentiated by chain rule, one can make the u become a possible substitution for the inner function, then convert this function into a form fully expressed in terms of u. After that, the calculation of the $\frac{du}{dx}$ and dx should be replaced by du. Currently, one need to integrate this new function of u. One thing needs to be noticed is that one need to use the substitutions u to rewrite the function, which depend on the relationship between x and u [5]. The special one is trigonometric substitutions. The function is not only can be replaced by single variable, but it also can be replaced by more complicated variable like $\sin u, \cos u$ and $\tan x$ such trigonometric substitutions. Depending on the trigonometric identities, one can transfer the function to easier form and integrate it. It is no different in principle than ordinary substitutions. When solving the integration which includes trigonometric function, the most useful method is to use trigonometric identities to make this function become easier, then it can be integrated directly with the help of [6]

$$sin^2 x = \frac{1-\cos 2x}{2}, cos^2 = \frac{1+\cos 2\mathrm{x}}{2} \tag{1}$$

and

$$sin^3 x = \frac{3\sin x - \sin 3x}{4}, cos^3\mathrm{x} = \frac{3\cos\mathrm{x} + \cos 3\mathrm{x}}{4}. \tag{2}$$

Integration by parts is to integral two functions when they are multiplied. Beginning at the product rule $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$, then this function can be changed to $f(x)g(x) = \int f'(x)\,g(x)dx + \int f(x)g'(x)dx$. Therefore, it is stated that [7]

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx\,. \tag{3}$$

Integration by parts is to turn one integral into another. Assume that $m = f(x)$ and $n = g(x)$ then $dm = f'(x)dx$ and $dn = g'(x)dx$ and

$$\int mdn = mn - \int ndn\,. \tag{4}$$

The numerator and denominator both are represented by polynomials, which can be called a rational function. If the denominator is $(ax+b)^n$, $u=ax+b$ can be a substitution.u^n is the denominator and $\frac{(m-b)}{a}$ is the expression of each x in the numerator, and $dx=\frac{dm}{a}$.

Finally, there is also a method called numerical integration. The main purpose of the integral is to find the area under the given function. The rectangles and trapezoids under this curve, so the obvious fact is that the trapezoids could give approximation of the integration of this curve [8]. Namely,

$$I=\left(\frac{f(x_0)}{2}+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{f(x_n)}{2}\right)\triangle x, \quad (5)$$

and this is known as the Trapezoid rule.

3. Applications

The first example is

$$I=\int\frac{5x+5\sqrt{x+2}}{\sqrt[3]{((x+2)^2)}}dx. \quad (6)$$

To calculate the integral, it is useful to divide this function into three parts, i.e., $5(\mathrm{x}+2)^{1/3}-\frac{8}{(\mathrm{x}+2)^{2/3}}+\frac{5}{(\mathrm{x}+2)^{1/6}}$. At this time, let $x+2=t$, then

$$\int\frac{5x+5\sqrt{x+2}}{\sqrt[3]{(x+2)^2}}dx=\int 5t^{1/3}-8t^{-2/3}+5t^{-1/6}dt=3.75t^{4/3}-24t^{1/3}+6t^{5/6}. \quad (7)$$

The second example is

$$I=\int_0^{\frac{\pi^3}{8}}dx\int_{\sqrt[3]{x}}^{\frac{\pi}{2}}dy\cos\frac{2x}{\pi y}. \quad (8)$$

Exchanging the positions of dy and dx and rewrite the boundary of this function as

$$I=\int_0^{\frac{\pi}{2}}dy\int_0^{y^3}dx\cos\frac{2x}{\pi y}. \quad (9)$$

After some calculation, it is found that

$$I=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}y\sin\left(\frac{2y^2}{\pi}\right)dy=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\sin\frac{2y^2}{\pi}d\left(\frac{2y^2}{\pi}\right)=\frac{\pi^2}{8}\left(-\cos\left(\frac{\pi}{2}+\cos 0\right)\right)=\frac{\pi^2}{8}. \quad (10)$$

The third example is [9]

$$I=\int_\infty^0\frac{x\ln x}{(x^2+1)^2}dx. \quad (11)$$

Divide it into two single functions, it is found that

$$I=\int_0^1\frac{x\ln x}{(x^2+1)^2}+\int_1^\infty\frac{x\ln x}{(x^2+1)^2}dx \quad (12)$$

Let $\mathrm{x}=1/t$, and the range of second integral change to 0 to 1. Then,

$$I=\int_0^1 \frac{x\ln x}{(x^2+1)^2}dx-\int_0^1 \frac{\ln t}{\left(\frac{1}{t^2}+1\right)^2 t^3}dt. \tag{13}$$

Times t in the denominator and nominator of second function at same time.

$$I=\int \frac{x\ln x}{(x^2+1)^2}dx-\int_0^1 \frac{\ln t}{\left(\frac{1}{t^2}+1\right)^2 t^4}. \tag{14}$$

As mentioned above $\mathrm{x}=1/t$, so one can use x to replace $1/t$. Therefore, it is found that

$$I=\int_0^1 \frac{\mathrm{x}\ln \mathrm{x}}{(\mathrm{x}^2+1)^2}\mathrm{dx}-\int_0^1 \frac{x\ln x}{(x^2+1)^2}dx=0. \tag{15}$$

The fourth example is

$$I=\int_0^\infty \frac{\sin(\tan x)}{x}dx. \tag{16}$$

Firstly, the function can be changed to the range of this integral, which is found to be

$$I=\int_0^\infty \frac{\sin(\tan x)}{dx}dx=\frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin(\tan x)}{x}dx. \tag{17}$$

Simplifying this integration, it is

$$I=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\tan x)\sum_{k=-\infty}^{+\infty}\frac{1}{x+k\pi}=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(\tan x)\left(\frac{1}{x}+2x\sum_{k=1}^{+\infty}\frac{1}{x^2-k^2\pi^2}\right)dx, \tag{18}$$

which equals to

$$I=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(\tan x)}{\tan x}dx=\frac{1}{2}\zeta\int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x^2+1)}dx. \tag{19}$$

Now, using the Cauchy-residue theorem, it is equivalent to

$$I=\frac{1}{2}\zeta\left[2\pi i\left(-\frac{1}{2}Res\, f(0)+Res\, f(i)\right)\right]=-\frac{\pi}{2}\frac{e+1}{e}. \tag{20}$$

The fifth example is [10]

$$I=\int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx \tag{21}$$

Let $f(a)=\int_0^1 \frac{\ln(1+ax)}{x(x^2+1)}dx$, then

$$f'(a)=\int_0^1 \frac{dx}{(1+ax)(1+x^2)}=\frac{\pi}{4(1+a^2)}-\frac{a\ln 2}{2(1+a^2)}+\frac{a\ln(1+a)}{1+a^2}. \tag{22}$$

Therefore,

$$I=\int_0^1 \frac{\ln(1+x)}{x(x^2+1)}dx=f(1)=\int_0^1 f'^{(a)}da=\frac{\pi^2}{16}-\frac{1}{4}(\ln 2)^2+\int_0^1 \frac{a\ln(1+a)}{1+a^2}da. \tag{23}$$

Let $g'(b)=\int_0^1 \frac{a^2}{(1+ba)(1+a^2)}da=\frac{\ln(1+b)}{b}-f'(b)$ and notice that $g(1)=\int_0^1 \frac{\ln(1+b)}{b}db-I=\frac{\pi^2}{12}-I$. So, the answer is $I=\frac{\pi^2}{16}-\frac{1}{4}(\ln 2)^2+\frac{\pi^2}{12}-I$, i.e.,

$$I=\frac{7\pi^2}{96}-\frac{\ln 2^2}{8}. \tag{24}$$

The sixth example is

$$I=\int_0^1\left(\frac{x^{q-1}}{1-ax}-\frac{x^{-q}}{a-x}\right)dx. \tag{25}$$

Firstly, this integral can be separated into two parts. Namely,

$$I=\int_0^1\left(\frac{x^{q-1}}{(1-ax)^{1-s}}-\frac{x^{-q}}{(a-x)^{1-s}}\right)dx=a^{-q}\int_0^a\frac{t^{q-1}}{(1-t)^{1-s}}dt-a^{-q+s}\int_0^{\frac{1}{a}}\frac{t^{-1}}{(1-t)^{1-s}}dt. \tag{26}$$

Let $s\to 0$, so Eq. (26) can be rewritten

$$I=\int_0^1\left(\frac{x^{q-1}}{1-ax}-\frac{x^{-q}}{a-x}\right)dx=a^{-q}\left(\int_0^a\frac{t^{q-1}}{1-t}dt-\int_0^{\frac{1}{a}}\frac{t^{-q}}{1-t}dt\right). \tag{27}$$

The part of the right bracket above takes a zero derivative of the parameter a and is therefore a constant, so it is found that

$$I=a^{-q}\left(\int_0^1\frac{t^{q-1}}{1-t}dt-\int_0^1\frac{t^{-q}}{1-t}dt\right)=a^{-q}\int_0^1\frac{t^{q-1}-t^{-q}}{1-t}dt=a^{-q}\pi\cot\pi q. \tag{28}$$

4. Conclusion

The integration has been invented long time ago, and people use it in many fields like in Economics. The integration can be used to find the maximum value and increasing rate in real world. This paper shows the different techniques to integrate functions. When solving integral problems, it is necessary to justify whether this integration can be solved by existing formulae. If it cannot be solved by fixed formula, one should think about any transformations about this function, and just integrate this function directly. If it also does not work, the substitution trick should be used. Using u to represent some complex function, then one can amend this function so the whole function will be written by u and dx should be replaced by du. The special one is trigonometric substitutions which can replace some complex variable like $\sin u$, $\cos u$ and $\tan u$. Integration by parts is to integrate one part of the whole function. Using trapezoid rule can work out the area under the curve, which called numerical integration. However, the numerical integration is only can be used in approximation of integral, which cannot obtain accurate results. All in all, the most important step is to figure out the best method to compute the integrals.

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