Basic Concept About the Integral and Its Basic Applications

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Abstract. Calculus is a breakthrough in human's development and its development is keeping making progresses. This paper mainly talks about the integral, which is a crucial component of calculus. The appearance of integral is much earlier than another part of calculus, the differentiation. The integral helps people to calculate the area, length, and volume of some irregular and sophisticated figures or objects that cannot be simply computed with limited existing tools. The integral appears in many forms, including simpler form and more sophisticated one. The more sophisticated form of integral may comprise two or more compound functions, so that it cannot be calculated simply by the principle of integral. In order to overcome such challenges, mathematicians later developed many specific methods, which are corresponding to some complex forms of integral respectively, such as u-substitution, inverse substitution, integral by parts, improper integral, residue theorem, and Cauchy integral formula. Also, for the examples in this essay, several methods mentioned upon are applied, so that some compound functions including logarithm, exponent, trigonometric function, and infinity are all successfully computed. This essay briefly concludes the history and importance of calculus and the functions of differentiation and integral, detailly introduces the property and operation of integral, provides some examples of integral, and talked about the principle and applications of specific methods of finding integral used in examples is this essay.

Keywords: Calculus, Definite integral, Indefinite integral, Integration by parts.

1. Introduction

Basically, the calculus is a branch of math study which mainly specializes functions about derivatives, continuity, integrals, infinity, and infinite series. The calculus consists of two parts: differentiation and integral, and this essay mainly talks about the integral. Differentiation basically refers to the instantaneous rate of change, which essentially is the slope of an infinitely small fraction of a curve of a function, and the slope of the tangent line of the curve of a function can be evaluated based on the derivative of that function [1]. Also, the derivative can be viewed as the velocity, and its second derivative, which is the derivative of that derivative in the context, refers to the acceleration in physics. Integral is commonly used to find the area under a fraction of curve and upon the x-axis. Using integral to find an area basically is to cut that area into many infinitely small rectangles and add them all. Integral can be further applied to evaluate the volume of an object produced by the spinning the curve of a function [2]. The integral also can be used to find the distance traveled of an object simply by finding the antiderivative of the its velocity and the pressure of a dam given its depth.

The invention of calculus is traced back to seventeenth centuries, and it was developed by English mathematician named Isaac Newton and Germany mathematician named Gottfried Wilhelm Leibniz independently. The calculus was originally called "Infinitesimal calculus". The development of calculus is mainly divided into three stages: ancient, medieval, and modern. The ancient development of calculus mostly occurred in Egypt and Babylonia, Greece, and China. The concepts of the computation of volume and area and the trapezoidal rule were initially introduced by the ancient Egyptian mathematician Moscow papyrus, which are the essence of integral [3]. However, these concepts introduced by Moscow papyrus did not have some reliable basics such as deduction and only was partially true, so the calculus in that era was just introduced instead of well established. In ancient Greece, the area and volume were successfully calculated by the mathematician Eudoxus, through the method of exhaustion, and this method was later developed further by Archimedes, the inventor of heuristics, a method that was similar with concept of integral and was incorporated in his work called The Quadrature of the Parabola, The Method, and On the Sphere and Cylinder. During that time, the infinitesimal was just a paradox, but later, the Greek mathematicians provided the geometric proof of infinitesimals, so that this proposition was judged as true. Eventually, Newton incorporated them into the framework of integral in calculus. In ancient China, mathematician Liu Hui invented the method of exhaustion and successfully utilized it to evaluate the area of a circle [4]. In the medieval stage, its development mainly occurred in middle eastern areas, India, and Europe. In the middle eastern areas, mathematician Hasan Ibn al-Haytham found a formula which specializes computing the sum of integral squares and fourth powers to calculate the area and volume of a paraboloid. In India, mathematicians in Kerala school of astronomy and mathematics identified some primary constituents of calculus and categorized them into derivative and integral. In Europe, Oxford Calculators and French collaborators proved the Merton mean speed theorem. In modern ages, Johannes Kepler defined the basic integral and introduced the concept of using Rieman-sum to calculate the area. In Europe, the concept of adequality, which is a method used for finding maximum, minimum, and tangent line to a curve, was created by several mathematicians including Isaac Barrow, René Descartes, Pierre de Fermat, Blaise Pascal, John Wallis [5].

The calculus is an essential branch of math, because it can actually be practiced in many fields of study. For instance, in engineering, calculus can be utilized to estimate the number of materials used for building a curved structure and the weight of that structure, in astronomy, people use it to detect the change of gravity of star, and in biology, researchers typically utilize the calculus to build the mathematical models to describe the growth and reproduction of organisms.

2. Integrals and Derivatives

Integral is divided into two modules, which are Indefinite Integral and Definite Integral. Indefinite Integral simply means the to find the “Antiderivative” of a function. To specifically explain, the Antiderivative is actually the inverse computation of the Derivative, using known Derivative of a function to find that function. The numeric expression evaluating a Definite Integral is like $\int f(x)\,dx$, where function f(x) is known Derivative of another function. Definite Integral evaluates the area between the interval of two points on a function and x axis. The expression of Definite Integral is like $\int_a^b f(x)dx$, where f(x) is a known Derivative of another function, and b, the upper limit, letter b in the expression, of this Definite Integral must be greater than letter a in the expression, which is the lower limit of this Definite Integral [6].

The essence of Indefinite Integral is to simply use the Derivative of a function to deduct its undifferentiated function. However, the essence of Definite Integral is relatively complicated. Normally, without calculus, if people want to compute such area, they typically break the area into several parts and calculate the sum of areas of several small part. However, such method, which is called Rieman sum, merely gets the approximation of the area, which is far less precise than using Definite Integral. The essence of Definite Integral is just making the value of Δx infinitely small and value of “n” infinitely big, which means to divide the area into infinite small parts. Such property is mathematically expressed as [7]

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{x=a}^{b} f(x)\,. \tag{1}$$

If a relation that the function F(x) is Derivative of the function f(x) is presented, by using the formula of evaluating Derivative, the value of function F(x) is evaluated from the value of function f(x), in contrast, in the calculation of Antiderivative, the value of function F(x) is typically provided at first, and based on function F(x), which is the Derivative of function f(x), the value of function f(x) is evaluated by oppositely use the formula of evaluating Derivative, which is known as “Antiderivative”. Therefore, while function F(x) is the Derivative of function f(x), function f(x) is the Antiderivative of function F(x). The Indefinite integral is written as $\int F(x)\,dx$=f(x). To calculate the Indefinite Integral of a function, the Derivative of that function should be presented, while the value of the function is still unknown, and then based on the value of the Derivative, the Antiderivative is evaluated, which is equivalent to the value of that function. The formula of finding the Antiderivative is that

$$\int ax^b dx = \frac{ax^{b+1}}{b+1} \tag{2}$$

To prove this formula, people can try to find the Derivative of $\frac{ax^{b+1}}{b+1}$. This differentiation process is that

$$\frac{d\left(\frac{ax^{b+1}}{b+1}\right)}{dx} = \frac{a(b+1)x^b}{b+1} = ax^b. \tag{3}$$

Therefore, this formula is absolutely correct. To calculate the Definite Integral of a function F(x) in calculus, the Antiderivative of F(x), which is expressed as function f(x), should be initially found, which uses the same function with computing Indefinite Integral, and then, the value b and a are plugged into the function f(x), which are f(b) and f(a). After that, f(a) is subtracted by f(b), and the product is the value of Definite Integral of F(x). Numeric expression of that process is [8]

$$\int_a^b F(x)\,dx = [f(x)]_a^b = f(b) - f(a). \tag{4}$$

3. Applications

3.1. Example I

The first example is

$$I = \int \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx\,. \tag{5}$$

First, according to the trigonometric relationship between $sin(x)$ and $cos(x)$ that $tan(x) = \frac{\sin x}{\cos x}$, the function can be expressed in terms of $tan(x)$. Namely,

$$I = \int \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int \frac{\sqrt{\tan x}}{\sqrt{\tan x} + 1} dx\,. \tag{6}$$

Using u to substitute the $\sqrt{\tan x}$, and then by using the u-substitution getting that: $\frac{du}{dx} = \frac{d\sqrt{\tan x}}{dx} = \frac{1}{2}(\tan x)^{-\frac{1}{2}}(1 + tan^2 x) = \frac{1+u^4}{2u}$. Therefore, $dx = \frac{du \cdot 2u}{1+u^4}$. To plug in the substitution, it is obtained that

$$I = \int \frac{2u^2}{(1+u)(1+u^4)} du \tag{7}$$

Then, by the partial fraction, the function can be divided into two parts:

$$I = \int \frac{1}{1+u} du + \int \frac{-u^3 + u^2 + u - 1}{1+u^4} du. \tag{8}$$

The next step is to figure out some simple antiderivative

$$I = \ln(1+u) - \frac{1}{4}\ln(1+u^4) + \int \frac{u^2 + u}{1+u^4} du. \tag{9}$$

And then, the unintegrated part is transferred to make it easier to be integrated

$$\int \frac{u^2+u}{1+u^4}du = \int \frac{d\left(u+\frac{1}{u}\right)}{\left(u+\frac{1}{u}\right)^2-1} + \frac{\frac{1}{2}\int d(u)^2}{1+u^{2^2}} = \frac{1}{2}\ln\frac{u^2-u+1}{u^2+u+1} + \frac{1}{2}\arctan u^2 + C. \tag{10}$$

The final result is obtained by plugging $\sqrt{\tan x}$ to u, which turns out to be

$$I = \ln(1+\sqrt{\tan x}) - \frac{1}{4}\ln(1+tan^2x) + \frac{1}{2}\ln\left(\frac{\tan x - \sqrt{\tan x}+1}{tan^2x+\sqrt{\tan x}+1}\right) + \frac{1}{2} + C\,. \tag{11}$$

3.2. Example II

The second example is [9]

$$I = \int \left(\frac{\arctan x}{x-\arctan x}\right)^2 dx \tag{12}$$

To begin with, since there is an inverse relationship in $arctan\ x$ and $tan\ x$, the x can be written as $x = \tan(\arctan x)$. Using u to substitute the $arctan\ x$, and then by using the u-substitution it is gotten that $\frac{d(\arctan x)}{dx} = \frac{1}{x^2+1}$. Therefore, $dx = x^2+1 = sec^2(\arctan x)$. By plugging in all these relationships into the u-substitution, it is obtained that

$$I = \int \frac{u^2}{(\tan u - u)^2} sec^2 u\, du = \int \frac{u^2}{(\sin u - u\cos u)^2} du\,. \tag{13}$$

Next, the funtion is divided into two parts by expanding and factorizing, and the antiderivative can be found

$$I = \int (\frac{-u}{\sin u})(\frac{-u\sin u}{(\sin u - u\cos u)^2} du = \frac{-u}{\sin u}\cdot\frac{1}{\sin u - u\cos u} - \frac{1}{\tan u} + C. \tag{14}$$

Eventually, by plugging in $\arctan x$ into u, it is obtained that

$$I = -\frac{1+x\arctan x}{x-\arctan x} + C. \tag{15}$$

3.3. Example III

The third example is

$$I = \int_0^\infty \frac{\sin x - x - x^3}{x^3(x^2+1)} dx\,. \tag{16}$$

To begin with, the function is divided into four parts through the partial fraction

$$I = \int_0^\infty \frac{\sin x}{x^3}dx - \int_0^\infty \frac{\sin x}{x}dx + \int_0^\infty \frac{x\sin x}{1+x^2}dx - \int_0^\infty \frac{1}{x^2}dx\,. \tag{17}$$

Next, some definite integrals of the function can be easily found through the residue theorem, e.g., $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$ and $\int_0^\infty \frac{x\sin x}{1+x^2}dx = \frac{\pi}{2e}$. Also, by using the partial fraction, it is obtained that $\int_0^\infty \frac{\sin x}{x^3}dx - \int_0^\infty \frac{1}{x^2}dx = \int_0^\infty \frac{\sin x - x}{x^3}dx$. Therefore,

$$I = \frac{\pi}{2} + \frac{\pi}{2e} + \int_0^\infty \frac{\sin x - x}{x^3}dx. \tag{18}$$

Next, the $\frac{1}{x^2}$ is substituted by u such that $\frac{du}{dx} = \frac{-2}{x^3}$ and $dx = \frac{x^3}{-2}du$. To apply the u-substitution, it is obtained that

$$I = \frac{\pi}{2} + \frac{\pi}{2e} + \int_0^{\infty} \left(\frac{\sin x - x}{x^3} \cdot \frac{x^3}{-2} \right) du = \frac{\pi}{2} + \frac{\pi}{2e} - \frac{1}{2} \int_0^{\infty} (\sin x - x) du . \tag{19}$$

By plugging in $\frac{1}{x^2}$ into the u, it is obtained that

$$I = \frac{\pi}{2} + \frac{\pi}{2e} + \frac{1}{2} \int_0^{\infty} \frac{\cos x - 1}{x^2} dx = \frac{\pi}{2} + \frac{\pi}{2e} + \frac{1}{2} \int_0^{\infty} \frac{\sin x}{x} dx . \tag{20}$$

Eventually, by using the residue theorem, one finds that the improper integral

$$I = \frac{\pi}{2} + \frac{\pi}{2e} - \frac{\pi}{4} = \frac{\pi}{2e} - \frac{3\pi}{4} . \tag{21}$$

3.4. Example IV

The fourth example is

$$I = \int_0^5 x e^{-\frac{5-x}{20}} dx . \tag{22}$$

To begin with, with the integration by parts in mind, one needs to make $\frac{dv}{dx} = e^{-\frac{5-x}{20}}$ and $u = x$. Therefore, by finding the antiderivative of $\frac{dv}{dx}$ and the derivative of u, it is gotten that $v = \int \frac{dv}{dx} = \int e^{-\frac{5-x}{20}} dx = 20 e^{-\frac{5-t}{20}}$ and $\frac{du}{dx} = 1, du = dx$. By applying the integration by parts, it is obtained that

$$I = \left[20 x e^{-\frac{5-t}{20}} - \int 20 e^{-\frac{5-t}{20}} dx \right]_0^5 = \left[20 x e^{-\frac{5-t}{20}} - 400 e^{-\frac{5-t}{20}} \right]_0^5 = 400 e^{-\frac{1}{4}} - 300 . \tag{23}$$

3.5. Example V

The fifth example is [10]

$$I = \int_{\pi}^{2\pi} cos^3 \frac{x}{2} sin^5 \frac{x}{2} dx . \tag{24}$$

The first step is to write the function into another form

$$I = \int_{\pi}^{2\pi} \left(1 - sin^2 \frac{x}{2} \right) sin^5 \frac{x}{2} \cos \frac{x}{2} dx \tag{25}$$

Next, the $\sin \frac{x}{2}$ is substituted with u by $\frac{du}{dx} = \frac{1}{2} \cos \frac{x}{2}$ and $dx = \frac{2du}{\cos \frac{x}{2}}$. Therfore,

$$I = \int_{\pi}^{2\pi} 2(1 - u^2) u^5 du = \int_{\pi}^{2\pi} (u^5 - u^7) du . \tag{26}$$

By using the antedirivative of the function with u, it is found that

$$I = \left[2 \left(\frac{1}{6} u^6 - \frac{1}{8} u^8 \right) \right]_{\pi}^{2\pi} = \left[2 \left(\frac{1}{6} sin^6 \frac{x}{2} - \frac{1}{8} sin^8 \frac{x}{2} \right) \right]_{\pi}^{2\pi} = -\frac{1}{12} . \tag{27}$$

4. Conclusion

In conclusion, all of examples provided in this essay has their own peculiarity except of merely finding the antiderivative. The first example and second example require the skills of using substitution and inverse differential property of trigonometric function. The third example practices the improper integral, while the fourth example is related to integration by parts and inverse differential property of logarithmic function. The fifth example utilizes the integration by parts as well as inverse differential property of trigonometric function. These methods used in examples are all common in problems of integral, and each of them corresponds to particular dilemmas that cannot be simply solved by antiderivative when evaluating an integral. When evaluating an integral $\int f(m(x))dx$, the substitution should be applied to change the variable $m(x)$ into u, so that the integral is simplified. Meanwhile, when an integral consists of a function with its variable, $\int f(x) \cdot f'(x)dx$, the substitution also can be applied to replace $f(x)$ with u, so that the integral is shifted and can be directly computed with antiderivative. The integration by parts is used when the function of the integral being evaluated combines the power function and exponential function, trigonometric function, or logarithmic function $\int f(x) \cdot m(x)\, dx$. In addition, when the function cannot be simplified by substitution, the integral can be directly computed by the formula of integration by parts. When the range of definite integral is infinity $\int_c^\infty f(x)\, dx$, the improper integral is applied to identify whether the integral is divergent or integrable. However, if the integral is judged as integrable, the improper integral can be further applied to evaluate it. To summarize, the study demonstrates that there are indeed many different approaches to calculate integrals.

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