Calculation of the Gaussian Integral by Real and Complex Approaches

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Abstract. The Gaussian integral is a very important integral in real analysis. It has many applications in various areas, including probability theory, statistical mechanics, and engineering. Nevertheless, the calculation of Gaussian integral is not easy and some tricks should be involved. To this end, this paper mainly focused on the Gaussian integral. There are three main parts about testifying of the Gaussian integrals, background and application. The first part is focused on the background and the meaning of Gaussian integral. The second part is mainly focused on three methods to testify Gaussian integral. Pertaining these methods, the first is about the real method and double integral, the second is about complex variables, and the third is about generalized integral. The last one is about to testify this integral by using normal integral under one variable integral. The third part of the paper is about the application of this integral. These include the extend of Gaussian integral, the general form of Gaussian which is more complex, and the Gaussian integral in thermodynamics, aiming to get general formulas on heat conduction.

Keywords: Gaussian integral; Definite integral; Heat conduction.

1. Introduction

Integration is the inverse of differentiation. The latter is the derivative of a function, while the former is to find the original function. In application, the integral function is not only be used in many area, it is widely used in summation and thus it is important in math. It can be used on many subjects such like math, physics, statistics or many other sciences and engineering major [1]. Many things on normal life need to use integral, such as displacement, acceleration, and velocity. Integral acceleration will become to the velocity, and integral velocity will become to the displacement. On other part, it also made people know some dangerous in daily life. For example, it is true that stand in a high place is dangerous is because of there is gravitational potential energy which can be calculated by the integral. So, come from this, one can know that integral is a very important part in our life. Also, it can be used to calculate the rate of change with the total number of changes. For example, it can get water into a tank, and calculate the rate of change of water per gallon per time in a function [2].

The Gaussian integral is a definite integral which seems like a bell curve. It can be used to calculate many things like distribution of some data number. The Gaussian integral is usually used on science, physics, statistics and probability theory. On physics, it used to calculate electrodynamics, thermodynamics or some other dynamics. On statistics, Gaussian integral can be used to calculate normal distribution and probability theory. The Gaussian integral is essentially the area under a Gaussian function. The normal distribution is a Gaussian probability distribution. The Gaussian integral will seem like [3]

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx, \int_0^{+\infty} e^{-a^{t^2}+bt}\,dt, \int_{-\infty}^{+\infty} x^{2n}e^{-x^2}\,dx. \tag{1}$$

The first one is a normal Gaussian integral. The expressions in Eq. (1) are a series of functions that take the shape of a "bell curve".

This paper will mainly discuss on testifying Gaussian integral and application of the Gaussian integral. There are many ways to testify Gaussian integral. On this passage, there will mainly testify this by four different methods. This paper will also focus on the Gaussian applied on thermodynamics of physics.

2. Original Gaussian Integral

2.1. The Real Method

The Gaussian integral is [4]

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\, dx. \tag{2}$$

One of the easiest ways to do so is by changing the one-dimensional integral to two dimensions. That is, the square of Eq. (2) can be written

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2}\, dx \int_{-\infty}^{+\infty} e^{-y^2}\, dy = \iint e^{-(x^2+y^2)}\, dx\, dy. \tag{3}$$

By setting $x = r\cos\theta$ and $y = r\sin\theta$, it is inferred that $(x^2+y^2) = r^2$, and $dx\, dy = r\, dr\, d\theta$. Therefore, the integral in Eq. (3) turns out to be

$$I^2 = \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-r^2} r\, dr\,. \tag{4}$$

By using of the fact that $\int_0^{2\pi} d\theta = 2\pi$, then

$$I^2 = \pi\int_0^{+\infty} e^{-r^2} r\, dr = -\pi\int_0^{-\infty} e^{-r^2}\, d(-r^2) = \pi. \tag{5}$$

Therefore, the value of the Gaussian integral is [5]

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{6}$$

2.2. Complex method

Let the integral [6]

$$\mathrm{T(s)} = \int_{-\infty}^{+\infty} x^{s-1} e^{x}\, dx\, (\mathrm{s} > 0), \tag{7}$$

which comes from the Remainder formula. If $s = 1/2$, then it is found that

$$\left[\mathrm{T}\left(\frac{1}{2}\right)\right]^2 = \frac{\pi}{\sin\frac{\pi}{2}} = \pi, \mathrm{T}\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{8}$$

Let $\mathrm{I} = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = 2\int_0^{+\infty} e^{-x^2}\, dx$. By setting $\mathrm{t} = x^2$, so, $\mathrm{x} = \sqrt{t},\ dx = \frac{1}{2\sqrt{t}} dt$ and thus

$$\mathrm{I} = 2\int_0^{+\infty} e^{t} \frac{1}{2\sqrt{t}} dt \int_0^{+\infty} t^{-\frac{1}{2}} e^{t}\, dt = \mathrm{T}\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{9}$$

Therefore, the Remainder formula can also be used to give the value in Eq. (6).

2.3. Generalized integral

First, there is a constant formula [7]

$$I=\int_{0}^{+\infty} e^{-\mathrm{a}t^2+bt}\,dt. \tag{10}$$

The divisor can be rewritten as $-\mathrm{a}t^2+t^2=-\mathrm{a}\left(t^2-\frac{b}{a}\mathrm{t}\right)=-\mathrm{a}\left(\mathrm{t}-\frac{b}{2a}\right)^2+\frac{b^2}{4a}$. So, the basic constant formula can be

$$I=\int_{0}^{+\infty} e^{-\mathrm{a}\left(\mathrm{t}-\frac{b}{2a}\right)^2+\frac{b^2}{4a}}\,dt\, e^{\frac{b^2}{4a}}\int_{0}^{\infty} e^{-[\sqrt{a}(t-\frac{b}{2a})]^2}\,dt=\sqrt{\frac{\pi}{4a}}e^{\frac{b^2}{4a}}. \tag{11}$$

Thus, there is an important formula such that

$$\int_{-\infty}^{+\infty} e^{-\mathrm{a}t^2\pm bt}\,dt=\sqrt{\frac{a}{\pi}}e^{\frac{b^2}{4a}}. \tag{12}$$

2.4. One Variable Calculus

First, for a common inequation $(\forall \mathrm{x})\geq 1+\mathrm{x}$, it is found that

$$1-\mathrm{x}^2\leq \mathrm{e}^{-\mathrm{x}^2}=\frac{1}{\mathrm{e}^{\mathrm{x}^2}}\leq\frac{1}{1+\mathrm{x}^2}. \tag{13}$$

This inequation set up on all $\mathrm{x}\in\mathrm{R}$, and for all $\mathrm{x}\in\mathrm{N}$. By using of the fact that $(1-x^2)^n\leq e^{-nx^2}\leq\frac{1}{(1+x^2)^n}$, it is concluded that [8]

$$\int_{0}^{1}(1-x^2)^n\,dx\leq\int_{0}^{+\infty} e^{-nx^2}\,dx\leq\int_{0}^{+\infty}\frac{dx}{(1+x^2)^n}. \tag{14}$$

On the left side integral, let $x=\cos t$, then $dx=-\sin t\,dt$. So,

$$L_n=\int_{0}^{1}(1-x^2)^n\,dx=\int_{0}^{\frac{\pi}{2}}(\sin t)^{2n+1}t\,dt=\frac{(2n)!!}{(2n+1)!!}. \tag{15}$$

On the right side integral, let $x=\tan t,\ dx=(\sec t)^2dt$. Thus,

$$R_n=\int_{0}^{+\infty}\frac{dx}{(1+x^2)^n}=\int_{0}^{\frac{\pi}{2}}(\cos t)^{2n-2}\,\mathrm{dt}=\frac{\pi}{2}\frac{(2\mathrm{n}-3)!!}{(2\mathrm{n}-2)!!}. \tag{16}$$

On the middle side integral, let x=$\frac{\sqrt{n}}{t}$, then $dx=\frac{dt}{\sqrt{n}}$. Therefore,

$$M_n=\int_{0}^{+\infty} e^{-nx^2}\,dx=\frac{1}{\sqrt{n}}\int_{0}^{+\infty} e^{-t^2}\,dt. \tag{17}$$

Summarizing those consequences, it can be found that [9]

$$\sqrt{n}*\frac{(2n)!!}{(2n+1)!!}\leq\int_{0}^{+\infty} e^{-t^2}\,dt\leq\sqrt{n}*\frac{\pi}{2}*\frac{(2\mathrm{n}-3)!!}{(2\mathrm{n}-2)!!}. \tag{18}$$

Let $\lim\limits_{n\to\infty}\sqrt{n}\frac{(2n)!!}{(2n+1)!!}=\frac{\sqrt{\pi}}{2}$ and $\lim\limits_{n\to\infty}\sqrt{n}\frac{\pi}{2}\frac{(2\mathrm{n}-3)!!}{(2\mathrm{n}-2)!!}=\frac{\sqrt{\pi}}{2}$. By using squeeze theorem, it is easy to find that $\int_{0}^{+\infty} e^{-t^2}\,dt=\frac{\sqrt{\pi}}{2}$, and thus $\int_{-\infty}^{+\infty} e^{-x^2}\,dx=2\int_{0}^{+\infty} e^{-x^2}\,dx=\sqrt{\pi}$.

3. Applications

3.1. Gaussian Integral with Power Function

Because e^{-x^2} is an even function, and $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$, $\int_{0}^{+\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. Now, there is a function $\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx = \mathrm{R}$ and $R^2 = -\pi\int_{0}^{-\infty} e^{-\alpha r^2}\,d(-\alpha r^2) = \frac{\pi}{4\alpha}$. On this time, $\mathrm{R} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$.

Let $\int_{-\infty}^{+\infty} x^1 e^{-x^2}\,dx$, and because it is an odd function, so it is equal to 0. So, on the function [10]

$$\int_{-\infty}^{+\infty} x^n e^{-x^2}\,dx \tag{19}$$

when n is odd, the function is equal to 0. On the other hand, when n is even, then write it on $2n$. It is calculated that

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}, \int_{-\infty}^{+\infty} x^4 e^{-x^2}\,dx = \frac{3}{4}\sqrt{\pi}. \tag{20}$$

Therefore, it is infeed that the formula holds

$$\int_{-\infty}^{+\infty} x^{2n} e^{-x^2}\,dx = \frac{(2n-1)!}{2^n}\sqrt{\pi}. \tag{21}$$

3.2. Application on Equation of Heat Conduction

Some preposition conditions are presented. Firstly, the initial heat distribution is zero. Secondly, the function of boundary endpoints over time is $f(x)$. Thirdly, the heat distribution is zero when position is infinity. Then, the equations of heat conduction are [11]

$$\begin{cases} \dfrac{\partial u}{\partial t} - a^2\nabla^2 u = 0 \\ u(x,0) = 0 \\ u(0,t) = f(t) \end{cases}. \tag{22}$$

By Laplace transform representation on $u(x,t)$ about time t, it is found that

$$L_t[u(x,t)](s) = \varphi(x,s). \tag{23}$$

The original function system become to

$$\begin{cases} s\varphi - a^2\varphi'' = 0 \\ \varphi(0,s) = F(s) \end{cases} \tag{24}$$

The first function can simplify to

$$\varphi'' - \frac{s}{a^2}\varphi = 0 \tag{25}$$

This is a second order homogeneous ordinary differential equation with constant coefficients, the solution of the function is

$$\varphi(\mathrm{x},\mathrm{s}) = \mathrm{A}(\mathrm{s})\mathrm{e}^{-\frac{\sqrt{\mathrm{s}}}{\mathrm{a}}\mathrm{x}} + \mathrm{B}(\mathrm{s})\mathrm{e}^{\frac{\sqrt{\mathrm{s}}}{\mathrm{a}}\mathrm{x}} \tag{26}$$

Because of the natural boundary condition. The second item should be zero. So $\varphi(x,s) = A(s)e^{-\frac{\sqrt{s}}{a}x}$. By substitution, it is found that $\varphi(0,s) = F(s)$ and $A(s) = F(s)$. So, $\varphi(0,s) = F(s)e^{-\frac{\sqrt{s}}{a}x}$. By inverse laplace transformation on $\varphi(x,s)$ about time t, it is found that

$$u(x,t) = f(t)\mathrm{erfc}'(\frac{x}{2a\sqrt{t}}) = f(t)\frac{x}{2at\sqrt{\pi t}}e^{-\frac{x^2}{4at}} = \frac{x}{2a\sqrt{\pi}}\int_0^t f(t-\tau)e^{-\frac{x^2}{4a\tau}}\frac{d\tau}{\tau\sqrt{\tau}}. \tag{27}$$

4. Conclusion

The paper first talks about the background of Gaussian and what will be introduced in this essay. Then, the author talks four methods to testify Gaussian integral. All of these methods are under the simple Gaussian and get the same answer of $\sqrt{\pi}$. On these methods, one variable calculus, generalized integral, real method and complex method are been used. The difficult point is usually happened on method of element changing and calculate limits. The author then uses element changing and simultaneous equations on many times. The next part of the essay introduced some application of Gaussian integral. The first one is to extend Gaussian integral and get extend normal equation of Gaussian integral, leading to the answer of extended Gaussian formula. In addition, the use of Gaussian integral in other areas like equation of heat conduction in physics. The most difficult point on these applications is Laplace transformation and inverse Laplace transformation. By this way, the author first uses Laplace transformation to change the form and simplify it, and then use inverse Laplace transformation to change the form and make the form of equation into a useful form on heat conduction of thermodynamics. To summarize, this study demonstrate that the Gaussial integral is a very useful integral in real analysis, and it has many applications in science.

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