Methods to Find the Closed form of $\zeta(2)$ and $\zeta(2n)$

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Abstract. Finding the closed form of the celebrated special function is not an easy task. For this purpose, this article deals with the Riemann zeta function. The Riemann zeta function is one of the most important functions in mathematics, and it mainly relates to the area of analytic number theory. Based on this function, the Riemann hypothesis was also proposed. This function is defined as $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, with $\mathrm{Re}(s) > 0$. In this article, several methods are provided to calculate specific values of Riemann zeta function. Specifically, the value when $s = 2$ and, more generally, when $s = 2n$. Finding the value of the function at two was used to be a world-class challenge, named Basel Problem. No one had solved this problem until Euler appeared, and since then many ways had appeared. In this article, two ways to solve the Basel problem are firstly provided. Afterwards, method involving Bernoulli numbers to calculate all the ζ-values at even arguments is also shown.

Keywords: Riemann zeta function; Basel problem; Bernoulli numbers; Closed form.

1. Introduction

The history of Riemann zeta function can be traced back to around 1350s. At that time, a mathematician called Nicole Oresme found and proved that the harmonic series, $\sum_{n=1}^{\infty} 1/n$, diverges. However, the harmonic series is just identical to $\zeta(1)$. In 1644, Pietro Mengoli posed the famous Basel problem, which was not the name before it was solved. The problem left open for many years, and it had welcomed its first solution by Leonhard Euler in the early 1700s, which is $\pi^2/6$. Even though Euler did not give the most rigorous prove, his methods and thoughts were technically ingenious and beautiful. In 1737, the Euler product formula was found by the same person, expressed as $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$. He also tried to calculate the values of $\zeta(-1)$, $\zeta(-2)$, and $\zeta(-3)$, and so on. What he did really engaged the progress of discovering about the zeta function. After the ear of Euler, Riemann also made big progresses and he managed to extend the study of zeta function to the complex field. The hypothesis claims that all the non-trivial zeros have the real part of $1/2$. Thousands of mathematical theorems were assuming that this hypothesis is correct, and many years passed since the hypothesis was invented, with no one successfully proved it.

Tracing back to $\zeta(2)$, Euler's method was to evaluate an integral and take its limit. Besides that, many methods were found. Apostol's method using only the knowledge in Higher Mathematics published in 1973 [1], Boo Rim Choe's method on American Mathematical Monthly in 1987 [2], and so on. Their efforts, by the way, highlight the importance of the problem. In this article, two specific ways to solve the problem are presented. Considering the importance of the function, a more general way to solve for all even values of Riemann zeta function is also discussed.

2. Methods to calculate $\zeta(2)$

2.1. Method 1: Improper Integral

The first method is similar to Euler's method, but they are totally different, and this method here would be easier to understand. To begin with, it is useful to convert the problem into another form. First, notice that when $0 < x < 1$, it is found that

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = \int \left[\sum_{n=1}^{\infty} x^{n-1}\right] dx = \int \frac{1}{1-x} dx = -\ln(1-x). \tag{1}$$

Then, the author considers the integral $\int_0^1 \frac{\ln(1-x)}{x} dx$. Substituting Eq. (1) into this integral, and it turns out that

$$\int_0^1 \frac{\ln(1-x)}{x} dx = -\int_0^1 \left[\sum_{n=1}^{\infty} \frac{1}{n} x^n\right] \cdot \frac{1}{x} dx = -\sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{2}$$

The right-hand side of this Eq. (2) is just $\zeta(2)$, so the problem is now changed to evaluate the right-hand side integral.

As this integral is difficult to deal with, other preparation is needed. In this method, another integral depending a parameter is constructed in order to help. After a few steps, the parameter can be replaced, and the solution will be derived. The integral is,

$$f(\alpha) = \int_0^1 \frac{\ln(x^2 - 2x\cos\alpha + 1)}{x} dx, \quad 0 \le \alpha \le 2\pi. \tag{3}$$

Notice the identity equation $(x^2 - 2x\, cos\, \alpha + 1)(x^2 + 2x\, cos\, \alpha + 1) = x^4 - 2x^2\, cos\, 2\alpha + 1$. This means that for this f, the values at $\frac{\alpha}{2}$, $\pi - \frac{\alpha}{2}$, and α can be related together as follows

$$f\left(\frac{\alpha}{2}\right) + f\left(\pi - \frac{\alpha}{2}\right) = \frac{1}{2} f(\alpha). \tag{4}$$

This is a functional equation, and its solution is also provided here [3]. To solve this equation, notice that f is twice continuously differentiable for all $\alpha \in [0, 2\pi]$. By differentiating the equation twice, it is found that $f''\left(\frac{\alpha}{2}\right) + f''\left(\pi - \frac{\alpha}{2}\right) = 2f''(\alpha)$. While f'' is continuous in the interval $[0,2\pi]$, $f''(\alpha)$ has a maximum value M and a minimum value m in its interval. Suppose that $f''(\alpha_0) = M$ for $\alpha \in [0, 2\pi]$, then it is arrived that

$$f''\left(\frac{a_0}{2}\right) + f''\left(\pi - \frac{a_0}{2}\right) = 2f''(a_0) = 2M. \tag{5}$$

However, since $f''\left(\frac{a_0}{2}\right) \le M$, $f''\left(\pi - \frac{a_0}{2}\right) \le M$, and $f''\left(\frac{a_0}{2}\right) + f''\left(\pi - \frac{a_0}{2}\right) \le 2M$, it can be deduced that the only way to satisfy these inequalities is $f''\left(\frac{a_0}{2}\right) = M = f''(a_0)$. By continuity it follows that

$$\lim_{n\to\infty} f''\left(\frac{a_0}{2^n}\right) = f''(0) = M. \tag{6}$$

From the other side, using similar ideas, it is easy to see that $f''(0) = m$, thus $M = m$, so that $f''(\alpha)$ is a constant function in $[0, 2\pi]$. Let $f''(\alpha) = a$. Then,

$$f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c, \tag{7}$$

where a, b, and c are constants. To determine the coefficients of this expression, what is needed is to substitute Eq. (7) into the original function Eq. (4), and to solve the derived equations. Afterall, the integral $f(\alpha)$ can be expressed as

$$f(\alpha) = -\frac{(\alpha - \pi)^2}{2} + \frac{\pi^2}{6}. \tag{8}$$

Once this Eq. (8) is determined, let $\alpha = 0$, and it shows that,

$$\int_0^1 \frac{l\,n(x^2 - 2x + 1)}{x} dx = 2\int_0^1 \frac{l\,n(1-x)}{x} = -\frac{\pi^2}{2} + \frac{\pi^2}{6}, \tag{9}$$

Which means that $\int_0^1 \frac{ln(1-x)}{x} = -\frac{\pi^2}{6}$. Bring the result of Eq. (9) back to Eq. (2), it is concluded that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = -\int_0^1 \frac{\ln(1-x)}{x} dx = \frac{\pi^2}{6} \tag{10}$$

Thus, the value of $\zeta(2)$ is derived.

2.2. Method 2: Infinite Product

First, the author focuses on the infinite product expansion [4] and power series [5] of $\frac{\sin x}{x}$, which is found to be $\frac{sin\,x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$ and $\frac{sin\,x}{x} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{(2k-1)!}$. This means that this infinite product can be equal to the infinite sum, which is [6]

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{(2k-1)!} \tag{11}$$

The next step is to take the natural logarithm for both sides, so that

$$\sum_{n=1}^{\infty}\ln\left(1 - \frac{x^2}{n^2\pi^2}\right) = \ln\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{(2k-1)!} \tag{12}$$

Notice that by using the power series of $\ln(1-x)$ and $\ln(x)$, both sides of Eq. (12) can be expanded as

$$\sum_{n=1}^{\infty}\ln\left(1 - \frac{x^2}{n^2\pi^2}\right) = \sum_{n=1}^{\infty}\left(-\frac{x^2}{n^2\pi^2} - \frac{1}{2}\frac{x^4}{n^4\pi^4} - \cdots\right) \tag{13}$$

and

$$\ln\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{(2k-1)!} = \left(-\frac{x^2}{6} + \frac{x^2}{120} - \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{6} + \frac{x^2}{120} - \cdots\right)^2 + \cdots. \tag{14}$$

As for the two expressions of Eq. (13) and Eq. (14), the x^2 terms can be taken out to form a more explicit equation $\sum_{n=1}^{\infty}\frac{x^2}{n^2\pi^2} = \frac{x^2}{6}$. The rest of the steps will be quite easy. By simplifying Eq (14), the final answer for the Basel problem is $\zeta(2\,) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

3. The Value of $\zeta(2n)$

The proof for $\zeta(2n)$ can also be started by looking at $\frac{\sin x}{x}$. The infinite product of it is already known: $\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$. Substitute s by πs, then [7]

$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right) \tag{15}$$

Take the logarithm of both sides, then

$$\log\frac{\sin \pi s}{\pi s} = \log\prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right) \tag{16}$$

and

$$\log \sin \pi s = \log \pi s + \sum_{k=1}^{\infty} \log\left(1 - \frac{s^2}{k^2}\right) \tag{17}$$

Differentiate both sides, the following equation is acquired [8]

$$\frac{d(\log \sin \pi s)}{ds} = \frac{\cos \pi s}{\sin \pi s}\pi = \frac{d}{ds}\left[\log \pi s + \sum_{k=1}^{\infty} \log\left(1 - \frac{s^2}{k^2}\right)\right] = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{1}{\left(1 - \frac{s^2}{k^2}\right)}\left(-\frac{2s}{k^2}\right). \tag{18}$$

By using a way to change Eq. (18) into another form, partial fraction can be used to decompose the summation symbol term, as follows

$$\pi \cot \pi s = \frac{1}{s} - \sum_{k=1}^{\infty} \frac{2s}{(k^2 - s^2)} = \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{s+k}. \tag{19}$$

Alternatively, another way to change the form of Eq. (18) is to consider $\frac{1}{1-s^2/k^2}$ as an infinite geometrical series with the first term being 1 and the common ratio being $\frac{s^2}{k^2}$ as $\left|\frac{s^2}{k^2}\right| < 1$ [9]. Therefore,

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{s^2}{k^2}\right)^n\left(-\frac{2s^2}{k^2}\right) = 1 - 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)s^{2n} = 1 - 2\sum_{n=1}^{\infty}\zeta(2n)s^{2n} \tag{20}$$

Notice that by using the Euler's Formula,

$$\pi s \cot \pi s = \pi s \frac{\cos \pi s}{\sin \pi s} = \pi s i \frac{e^{i\pi s} + e^{-i\pi s}}{e^{i\pi s} - e^{-i\pi s}} = \pi s i \frac{e^{2i\pi s} + 1}{e^{2i\pi s} - 1} = \pi s i + \frac{2\pi s i}{e^{2i\pi s} - 1} \tag{21}$$

Before proceeding further, definition of Bernoulli's Numbers B_n is helpful, which is defined as [10]

$$\frac{s}{e^s - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} s^n . \tag{22}$$

Now, substituting Eq. (22) into Eq. (21), it is found that

$$\pi s \cot \pi s = \pi s i + \frac{2\pi s i}{e^{2i\pi s} - 1} = \pi s i + \sum_{n=0}^{\infty} \frac{B_n}{n!}(2\pi s i)^n = \pi s i + \sum_{n=0}^{\infty} \frac{B_n}{n!}(2\pi s i)^n . \tag{23}$$

Since $B_0 = 1, B_1 = -\frac{1}{2}$, then

$$\pi s \cot \pi s = \pi s i + \frac{B_0}{0!} + \frac{B_1}{1!}(2\pi s i) - 2\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)\frac{B_n}{n!}(2\pi s i)^n = 1 - 2\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)\frac{B_n}{n!}(2\pi s i)^n. \tag{24}$$

As all the odd positions except "1" of Bernoulli Numbers have the value of 0, thus

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\frac{B_{2n}}{2n!}(2\pi s i)^{2n} = 1 - 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2 \cdot 2n!}s^{2n} . \tag{25}$$

At the same time, from Eq. (20) it is known that $\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty}\zeta(2n)s^{2n}$. The result can be derived by corresponding items in the same position in the two equations, which is

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2 \cdot 2n!}. \tag{26}$$

4. Conclusion

To conclude, this article provides ways to solve for $\zeta(2)$ and $\zeta(2n)$, which helps people to know more about Riemann zeta function. There are numerous ways to prove the results. However, the simplicity of the proof is also important. A good solution usually owns its beauty, but also brings profound mathematical achievements clearly to more people. Methods provided in this article can be understood only using the knowledge from advanced mathematics. These methods can be used to expand ideas about Riemann zeta function. The importance of Riemann zeta function embodies in many aspects of the number theory area, and the most essential unproved challenge would probably be the Riemann hypothesis. However, it is not only of that, yet a general formula for all the odd numbers of zeta function is still not found. There is not much understanding of this function, and people can only gradually broaden people's understanding of it. Changing a problem to another one is a common trick while doing math problems. It converts the sum into an integral by using the property of integral itself, and uses the idea of geometric series to simplify the expression. However, the other method is different. The construction and discovery for tools or function to help solve the problem can be difficult to complete. As a result, the exploration of this function and even the progress of mathematics have a long way to go.

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