Formalism in the History of Philosophical Mathematics: Beyond Logicism and Intuitionism

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Abstract. An examination of the similarities and differences between these three schools may shed light on the richness and variety of mathematical philosophies, as well as the significance of such philosophies to the comprehension and growth of mathematics. Formalism, logicism, and intuitionism are the names of these schools. The purpose of this study is to investigate the three primary schools of thought within the philosophy of mathematics, examine the many ways in which these schools interpret and approach mathematics, and investigate the areas in which these schools share and diverge in their perspectives. All three schools make an effort to present mathematics in a manner that is rigorous and built on a strong foundation, but each of them must also contend with its unique obstacles and challenges. The transcendence of formalism regarding the other two will be shown throughout this investigation with the use of instances such as Godel's incompleteness theorem and infinitesimal quantities.

Keywords: Mathematical Philosophy; Formalism; Logicism; Intuitionism.

1. Introduction

Mathematics has the distinction of being one of the most ancient, exact, and universally applicable kinds of human knowledge. Its significance extends across a multitude of disciplines, including the natural sciences, engineering technologies, and social sciences [1-3]. Nevertheless, the precise definition of mathematics remains a subject of inquiry. What are the processes involved in the acquisition of mathematical knowledge? What is the ontological status of mathematical objects? What is the methodology for establishing mathematical truth? The inquiries in question have been a source of concern for scholars in the fields of mathematics and philosophy, and serve as the primary subjects of investigation within the discipline of philosophy of mathematics [4].

Philosophy of mathematics is the branch of philosophy that studies the nature, foundations, methods, and meaning of mathematics, and which attempts to answer the above questions and, in so doing, to account for the place of mathematics in the whole enterprise of reason [4, 5]. Philosophy of mathematics is concerned not only with mathematics itself but also with its relation to other fields (e.g., logic, language, epistemology, metaphysics, etc.). Philosophy of mathematics helps us to gain a deeper understanding of what mathematics is and what it is not, as well as the value and influence of mathematics in human culture.

In this study, the main research routine will look at the three main schools of philosophy of mathematics, analyze their different understandings and positions on mathematics, and explore the points of commonality and conflict between them. All three schools try to provide a solid foundation and a rigorous approach to mathematics, but they also face their difficulties and challenges. This study will use Gödel's incompleteness theorem and infinitesimal quantities as examples to point out the transcendence of formalism about the remaining two. The aim is to demonstrate the complexity and diversity of the philosophy of mathematics and its importance for the understanding and development of mathematics through a comparative analysis of these three schools.

2. Philosophical Foundations of Mathematics

Philosophy of mathematics involves the exploration of questions about the nature of mathematics, truth, existence, and the nature of mathematical objects [6]. There are several different schools or perspectives in this field, the three main ones being logicalism, formalism, and intuitionism [7]. These schools have sparked many interesting and insightful debates in the field of philosophy of mathematics, exploring the nature of mathematics, truth, and the relationship between mathematics and reality.

2.1. Logicism

Logicism emphasizes the foundational, rigorous, and universal nature of logic and mathematics. Its main point is that all mathematical truths can be proved by logical reasoning and that these truths are absolute and universal and do not depend on concrete experience or intuition [8]. Logicism seeks a pure and flawless foundation for mathematics and attempts to establish the unity and universality of mathematics through a formalized system of logic.

Representative figures of logicism include Frege, Russell, and Whitehead. They were committed to adopting the language and methods of logic to construct the foundational system of mathematics. Frege pioneered a new system of logic, predicate logic, which was more capable of expressing complex linguistic structures and mathematical concepts than traditional propositional logic. In addition, Frege introduced the distinction between sense and denotation to explain problems in language involving reference and meaning. Russell and Whitehead co-authored Principia Mathematica, a book that is considered the cornerstone of modern mathematical logic. In the book, they used type theory to solve known morphological and semantic paradoxes such as Russell's paradox. In addition, they developed the core idea of logicalism, which holds that the basic ideas of mathematics, such as natural numbers, real numbers, and sets, can be defined through logical concepts. Although Logicism had some successes in the early 20th century, the absolutist view of Logicism was gradually questioned in the context of Gödel's incompleteness theorem. Over time, people began to recognize the complexity and diversity of mathematics, prompting them to explore other schools of thought to better explain the nature of mathematics and knowledge.

2.2. Formalism

Formalism emphasizes that the focus of mathematics should be on formal systems and symbolic operations rather than on the specific interpretation or practical meaning of mathematical objects [9]. The main idea of formalism is that mathematics can be viewed as a formal system in which rules of reasoning and symbolic manipulation are its central components. An iconic figure of formalism is Hilbert, who worked to construct consistency and completeness in mathematics using a finite number of symbols and rules. He famously formulated mathematical problems that inspired developments and innovations in the field of mathematics in the 20th century. These problems span a wide range of disciplines, including set theory, logic, algebra, geometry, topology, analysis, and physics, some of them have been solved, some are unresolved, and the remainder are unsolvable. In addition, Hilbert worked with his students to establish several important formal systems, including Hilbert spaces, Hilbert axioms, and Hilbert operators. These formal systems have a wide range of applications in modern mathematics and physics, and play an important role, for example, in the fields of quantum mechanics, generalized function analysis, and spectral theory.

Although formalism provides a clear framework for dealing with the foundations and derivations of mathematics, some argue that ignoring the actual meanings of mathematical objects may lead to a kind of nihilism that is divorced from practical applications. In addition, formalism may encounter difficulties in dealing with some philosophical issues, such as how to explain the validity of mathematics and the process of generating mathematical concepts. That is why it would not be an easy task to say that formalism has already overlapped with the other schools of mathematics thoughts.

2.3. Intuitionism

Intuitionism is a philosophical view of mathematics, as opposed to formalism and logic, which emphasizes that the process of constructing mathematics should be based on intuition and human intuition, rather than relying solely on formal derivations and symbolic manipulations [10]. The central idea of intuitionism is that the truths of mathematics are arrived at as a result of human intuition and reflection, not merely through the derivation of the rules of a formal system.

Representatives of intuitionism include Brouwer, Kleeney, and Dummett. Brouwer initiated intuitionistic logic, a system of logic that differs from traditional classical logic in that it denies the universal applicability of the law of exclusion and the law of double negation, and recognizes only constructible proofs of existence and feasibility. Kleeney advanced intuitionistic arithmetic, a theory of arithmetic based on intuitionistic logic that avoids the use of infinite sets and infinite processes, and instead uses a finite recursive method to define natural numbers and arithmetic operations. Dammit, on the other hand, advanced intuitionistic type theory, a type theory based on intuitionistic logic that unifies mathematical objects and proofs into types and terms and introduces dependent types to express more complex mathematical structures. Dammit also proposed intuitionistic set theory, a theory of sets built on intuitionistic type theory that avoids the use of the axioms of infinity and the axiom of choice, and instead employs a finite constructive approach to defining sets and set operations.

Intuitionism generated some discussion and controversy in the early and mid-twentieth century. While the intuitionist perspective emphasized in some ways the importance of human creativity and intuitive thinking, it was also criticized by formalist and logical perspectives that placed greater emphasis on strict formalization and logical derivation.

3. Between Formalism and Logicism

Both schools emphasize the inherently logical and systematic nature of mathematics. Formalism believes that mathematical reasoning and manipulation should be based on strict rules, while logicalism bases mathematics on a precise system of logic. Both seek mathematical rigor, focusing on the completeness and consistency of proofs and avoiding contradictions and ambiguities.

In addition, both formalism and logic seek to capture the essence of mathematics through formal systems. Formalism tends to view mathematics as a symbolic game, focusing on symbolic transformations and rules of derivation, and emphasizing syntactic structure. Logicism, on the other hand, puts more emphasis on the semantic connotations of mathematics and the relationship between the interpretation of logical systems and mathematical concepts.

For the possible conflicts, formalism, and logicism have very different views about the nature of mathematical objects, how the foundations of mathematics are constructed, and the nature of mathematical truth and proof. The debate between these two views has enriched discussions in the philosophy of mathematics and has influenced the understanding of the nature of mathematical truth and proof. Formalism emphasizes the mechanical process of proof, while logicism emphasizes logic and positivity. That is, formalism sees mathematical precision as founded on paper, while logicalism sees this as founded on logic.

Despite their differences, they have jointly promoted the development of the philosophy of mathematics. By exploring formalism and logicism, people have thought more deeply about the nature and meaning of mathematics, thus promoting the intersection of mathematics and philosophy and enriching people's understanding of mathematics.

4. Formalism and Intuitionism

Similar to the comparison made above, both these schools are concerned with the foundations and rationality of mathematics. Formalism, while emphasizing formal structure, also believes that mathematical objects and concepts should be introduced with some inherent logical foundation. Intuitionism, on the other hand, believes that the reality of mathematics should be based on our intuition and experience, and emphasizes that mathematics should be compatible with human cognition. Furthermore, both formalism and intuitionism involve the philosophical foundations of mathematics. Formalism emphasizes the system of symbols and rules of mathematics and tries to capture the essence of mathematics through formal systems. Intuitionism emphasizes that mathematics is a product of human thinking and that the reality of mathematics should be influenced by human intuition and intuitive concepts.

There are significant differences between formalism and intuitionism regarding the sources of mathematical precision and validity. These differences represent the different views of the two on the nature of mathematical objects, the nature of proofs, and the foundations of mathematical reasoning. The differences between the two lie primarily in the way mathematical truths and proofs are understood. Formalism views mathematics as an abstract symbolic manipulation and logical deduction, emphasizing the rules and mechanical derivation of formal systems. Intuitionism, on the other hand, sees mathematics as based on human intuitive understanding, emphasizing the reality of mathematical objects and the intelligibility of proofs. To summarize, formalism believes that the precise validity of mathematics lies on paper, whereas intuitionism believes it is in human reason.

The opposition and debate between these two views enriches the discussion of the philosophy of mathematics and reflects different understandings of the nature of mathematics.

5. The transcendence of formalism in the other two

5.1. Gödel's Incompleteness Theorem as a Challenge to Logicism

Gödel's incompleteness theorems are an important set of mathematical theorems formulated by the Austrian mathematician Kurt Gödel at the beginning of the twentieth century, and they have had a profound impact in the fields of formal and mathematical logic. These theorems show that in some particular formal system, there exist statements whose truth cannot be proved or disproved by the rules and reasoning of that system. First Incompleteness Theorem states that in any sufficiently powerful system of arithmetic forms of natural numbers, there exist some statements that can be shown to be neither true nor false. In other words, there are propositions within these systems that cannot be made definitively true or false by the rules and reasoning methods of the system itself.

The second Incompleteness Theorem states that if a system of natural number arithmetic forms is consistent (i.e., does not lead to both true and false statements), then it is not possible to prove its consistency within that system. This means that if a system is strong enough to prove its consistency, then it must be inconsistent.

The proofs of these theorems are based on Gödelian constructions involving the encoding of natural numbers and the formal representation of statements. The discovery of these theorems had a profound impact on the study of the foundations of mathematics and formal logic, revealing the limitations within formal systems and the difficulty of not being able to fully capture mathematical truths.

The proof of Gödel's incompleteness theorem, on the other hand, is relatively complex, involving concepts such as formal logic, self-reference, and encoding. The idea of the proof is briefly summarized here.

The use of Peano arithmetic is used to describe the basic properties and rules of operation of the natural numbers, and it is assumed that this system is consistent, i.e., it does not lead to contradictory conclusions. Gödel's idea of the proof is based on the construction of a statement about the natural numbers that express the meaning of "I cannot be proved true in this system", a statement that can be represented by itself. This notion of self-reference and self-reference is the key to the proof. The distinction between propositions whose "truth-value is true" and propositions whose "meaning is true" is utilized to construct propositions whose meaning is true but whose truth-value is not provable, while avoiding the dilemma of falling into paradox. It should be noted that propositions in formal logic do not have meanings in themselves. Propositions have only truth values, not meanings. Axiomatic propositions have true values and other propositions have true values if and only if they are provable; truth values are false if and only if their non-propositions are provable. With proper encoding, it is possible to construct a statement G within the system called a "Gödel sentence", which says "In this system, G cannot be proved to be true".

This means that if the system can prove that G is true, then the proof itself implies that G is true, thus creating a contradiction. Similarly, if the system can prove that G is false, then a contradiction arises. Thus, if the system is consistent, then it cannot prove that G is true or false. This shows the existence of a statement (i.e., G) that is neither provably true nor provably false in a sufficiently powerful formal system of natural number arithmetic, i.e., the content of the First Incompleteness Theorem.

The basic idea of the proof of the Second Incompleteness Theorem is constructing a statement about the self-consistency of the system and showing that this would lead to a contradiction if the system could prove the statement. The specific details involve the representation and encoding of the system and how to construct a statement about its consistency.

5.2. Challenges awaiting Logicism

The logicist view that all mathematical propositions can be deduced from a simple set of axioms is shown to be false by Gödel's incompleteness theorem. Gödel's First Incompleteness Theorem states that there are true propositions in any sufficiently powerful formal system that cannot be proved within that system. Gödel's Second Incompleteness Theorem, on the other hand, states that no sufficiently powerful formal system can prove its consistency. These theorems show that the goals of logicism are impossible to achieve because there will always be true propositions that cannot be proved within a formal system.

Gödel's theorem proves that in a sufficiently powerful system of mathematical axioms, there must exist propositions that cannot be proved, which goes against the goal of completeness that logicism tries to establish.

Besides, incompleteness is just one main challenge for logicism. Self-referentiality is the idea that an utterance, proposition, or system describes or states itself in some sense, which could cause another problem. Self-referentiality has important applications in logic, mathematics, and philosophy, where it can be used to construct paradoxes and solve esoteric problems. A paradox is a proposition or situation that seems contradictory or absurd. Paradoxes usually challenge our understanding and perception of something and prompt us to rethink things. There are many types of paradoxes, some of which arise due to self-referentiality, such as the famous Liar's Paradox and Russell's Paradox. These paradoxes show that contradictions or inconsistencies may arise when a statement or system describes itself.

The proof of Gödel's incompleteness theorem utilizes the concepts of self-referentiality and paradox. Gödel constructs a statement that in some sense states that it cannot be proved within a given formal system. This statement is analogous to the famous "this statement is false" paradox, because if it is true, then it cannot be proved, but if it can be proved, then it must be false.

Gödel does this by encoding statements within the formal system as numbers and then constructing a statement that states that the statement represented by its encoded number cannot be proved. The fact that this statement can neither be proved nor falsified within the formal system shows that the formal system is incomplete.

To conclude, this creates a great challenge to logicism, which holds that all mathematical propositions can be deduced from a simple set of axioms. Gödel's Incompleteness Theorem showed that this view is false, as there will always be true propositions that cannot be proved within a formal system. This prompted mathematicians to rethink the foundations of mathematics and to accept that some propositions may never be proved or falsified.

A determinability problem is a question, which would render the issue of adjudicability, of whether there exists an algorithm for a problem that can determine in finite time whether any given input satisfies a certain condition. In mathematics, this usually refers to whether a proposition can be proved or falsified in a given formal system.

Gödel's Incompleteness Theorem shows that in any sufficiently powerful formal system, there exist some true propositions that cannot be proved within that system. This means that, for these propositions, there does not exist an algorithm that can determine whether they can be proved or falsified within a given formal system. Thus, Gödel's incompleteness theorem suggests that the goal of logicism is impossible to achieve because there will always be true propositions that cannot be proved or falsified within a formal system.

Gödel's proof does not specify a specific formal system but uses a general approach. This implies that consistency may vary between different formal systems. Logicism needs to deal with how to ensure the consistency of the chosen formal system and how to deal with possible problems of consistency. The incompleteness theorem revealed the inherent limitations of formal systems, which cannot simultaneously satisfy the three goals of completeness, consistency, and decidability. These discoveries have profoundly affected the fields of mathematics and philosophy, and have led to a rethinking of the foundations of mathematics and of formal ways of reasoning.

5.3. Difficulties with the concept of infinitesimal quantities

In traditional analytic mathematics, infinitesimal quantities are a concept used in calculus to describe limits. However, intuitionism has difficulties with infinitesimals. This is because intuitionism questions the existence and reliability of the concept of infinity and assumes that only finite and constructible objects can be discussed, and thus infinite quantities cannot be truly understood or manipulated.

Infinitesimal quantities are intuitively more difficult to understand because they are quantities infinitely close to zero but not equal to zero. Intuitionism refuses to recognize the abstract notion of real infinity and does not regard the notion of infinity, the set of all natural numbers, as an entity. Intuitionism emphasizes that only objects that can be constructed in finite steps are truly existent. Thus, intuitionism questions the existence of infinitesimals because it believes that infinitesimals cannot be constructed.

Intuitionism's problematic questioning of the operation of infinity stems primarily from its rigorous demands on mathematics and reasoning, as well as its skepticism of infinite processes. Infinitesimals are often used in traditional mathematics to perform various calculations and derivations. However, intuitionism holds that people can only handle a finite number of steps in an operation and cannot perform infinite operations. This is because infinite operations involve consideration of an infinite number of steps, and human cognitive and reasoning abilities may become unreliable and prone to doubt and error when dealing with an infinite number of steps. And the infinity operation does not have a direct counterpart in the real world, because things and processes in the real world are usually exhaustive. For intuitionism, a reliable foundation for mathematical concepts can only be established through an exhaustive and explicit process of operations and constructions.

Introducing infinitesimals into the system of intuitionism leads to paradoxes, such as Beckley's paradox, i.e., "Is an infinitesimal quantity 0 or not?". For practical applications, it must be both 0 and not 0. But in the intuitionist view, this certainly belongs to the paradox. The Abbott paradox is also a classic example of a paradox that can help illustrate the problems these concepts can raise. It is a paradox involving infinite division in dynamic motion. The context of the paradox is that Achilles and a tortoise are engaged in a race, and Achilles initially starts by giving the tortoise a starting lead distance. By the time Achilles has traveled this distance, the tortoise has advanced a short distance. By the time Achilles finishes this distance, the tortoise has advanced a shorter distance. This process seems to go on and on, splitting the distance infinitely, with Achilles never seeming to catch up with the tortoise. This represents some of the unusual results that logic can produce when an infinite split is performed.

Intuitionism raises legitimate questions about the concept of infinity, but the concept of infinity in traditional mathematics is undoubtedly successful in practical applications and provides a useful framework for understanding the natural world and solving problems. Therefore, intuitionism should look for solutions in addition to questioning.

5.4. Advantages and methods of formalism in solving such problems

Formalism treats infinitesimal quantities, infinitely large quantities, etc., as virtual symbols, which are not subjected to strict actual numerical manipulation but are manipulated according to certain rules. For example, an infinitesimal quantity is denoted as dx and an infinitesimal quantity as ∞, which are then derived and computed using some manipulation rules. This at the same time allows some calculus operations to be carried out directly on the symbolic level without involving the concept of limit too much.

In formalism, symbolic and formal operations can be performed directly without worrying about the practicality of these operations. For example, it is possible to perform algebraic operations on infinitesimals in the course of calculations without having to consider their practical feasibility.

Formalism extends the notion of a limit to more general cases, allowing mathematicians to deal with limits in extended mathematical structures outside the traditional domain of real numbers, such as hyperreal numbers, hyper infinitesimals, and so on. This contributes to a broader understanding of the nature of limits. Thanks to the fact that formalism provides strict definitions and operational rules for infinitesimals and infinitesimals, this allows mathematicians to operate on them formally. It also helps to deal with situations that traditionally tend to be confusing, such as cases where the numerator and denominator both tend to be zero.

Nonstandard analysis is another analytical method in mathematics that extends and complements traditional real analysis and calculus, aiming at solving problems that are difficult to deal with by traditional methods, especially cases involving infinitesimals and infinities. Its central idea is to introduce the concepts of infinitesimals and infinities, but unlike the traditional concept of limit, it does not need to rely on the convergence of a sequence or function. In non-standard analysis, the domain of the real numbers can be extended to a larger domain that includes the real numbers and elements such as infinitesimals and infinities, so that infinitesimals and infinities can be regarded as "additions" to the real numbers.

Formalism solves infinitesimal problems by introducing rigorous definitions and axioms. For example, the ε-δ definition of a limit is used to define derivatives and integrals in calculus. This approach ensures consistency of derivation. It is this that non-standard analysis is used to construct analytics.

For imaginary numbers and complex functions, formalism is doing better. It usually takes a more abstract approach when dealing with imaginary numbers and functions of a complex variable, emphasizing form and structure without having to focus too much on the actual meaning. In the formalist view, imaginary and complex numbers can be regarded as abstract mathematical objects that do not necessarily need to have a physical or geometric interpretation in practice. The imaginary unit i can be viewed as a symbol that satisfies i^2 = -1. Complex numbers can be expressed in the form a + bi, where a and b are real numbers. The formalism emphasizes the algebraic properties of complex numbers and the rules of arithmetic, making it possible to treat complex numbers as ordinary algebraic expressions in calculations.

However, with the development of the theory of quantum mechanics, the quantum theory based on real numbers has been falsifiable. Theoretically, the Schrödinger equation and Heisenberg's pairwise relation, which are the cornerstones of quantum mechanics, are built on the concept of complex numbers. Experimentally, the real and imaginary parts have also been detected in the wave function today. Thus complex numbers can be regarded as detectable physical facts. However, it is undeniable that formalism is still superior to intuitionism in the operation of complex numbers, and the form of complex numbers in formalism is easier to understand.

In dealing with functions of a complex variable, formalism takes a similar approach. It generalizes the operations on functions of a complex variable by introducing strict rules of arithmetic, such as the use of the Cauchy-Riemann equations for determining the differentiability of a function of a complex variable. For example, when calculating derivatives or integrals, mathematicians can treat functions of a complex variable as symbolic expressions and apply the rules of algebra and formal arithmetic to their derivations. The formalist approach permits freer operations in the complex plane, including analytic, integral, and series expansions of functions of a complex variable. These are difficult to accomplish with intuitionism, so at this level, formalism is superior to intuitionism.

6. The limits of formalism

Undeniably, formalism also has certain limitations. For example, the formalist view of mathematics ignores the true nature of mathematics, treating it merely as a set of symbols and rules and ignoring the profound truth and meaning it contains. This view has led to a gradual disconnect between mathematics and the real world, and mathematics has become isolated from practical applications, seeming to be only a kind of abstract game that lacks in-depth insight into practical problems.

In contrast, two conceptions of mathematics, logicalism and intuitionism, attempt to bridge this disconnect. Logicism associates mathematics with rigorous logic and emphasizes that mathematics is based on a series of rigorous reasoning and proofs, thus making the truth of mathematics more solid and reliable. Logicism attempts to construct the entire mathematical system in terms of a logical system, which enhances the credibility of mathematics by ensuring the consistency and integrity of mathematical conclusions through formalized derivations. On the other hand, intuitionism emphasizes that mathematics is an activity that arises directly from human intuition and gut feeling and is more concerned with the thought processes and insights behind mathematics. Intuitionism believes that mathematics is not just about the manipulation of a bunch of symbols but about the way of observing, discovering, and understanding the real world. By incorporating human intuition and perception into mathematics, intuitionism attempts to bring mathematics closer to the nature of human thinking, thus making it more practically applicable and philosophically meaningful.

Taken together, mathematics as a science is more than just a collection of abstract symbols; it is a way of discovering, exploring, and understanding the world. While emphasizing the universal regularities and natural phenomena of mathematics, the two conceptions of mathematics, logicalism and intuitionism, also endow mathematics with more beauty and inspiration. By incorporating human thinking ability and creativity into mathematics, mathematics can burst into a richer light through the interweaving of reason and emotion.

7. Conclusions

This article explores the transcendence of formalism over logicism and intuitionism from the perspective of the philosophy of mathematics. Formalism can accommodate a wide range of mathematical theories without being affected by Gödel's incompleteness theorem, whereas logicalism is in trouble because of its inability to prove the completeness and consistency of its formal system. Formalism was able to deal with the concept of infinitesimals, whereas intuitionism abandoned analytic development because it could not accept the concept of infinitesimals. Formalism is not the perfect final answer today either, but rather a view of mathematics that complements logic and intuitionism. Future in-depth research in this area will still need to consider the strengths and weaknesses of all three schools simultaneously, in search of a more comprehensive, balanced, and profound philosophy of mathematics.

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