# Integration and Its Applications on Integral, Limit, and Beyond

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**Abstract.** The definite integral is a very important concept in real analysis and it has many applications in diverse disciplines. For example, the area under the curve can be calculated by cutting and approximating the curve, and the arc length of the curve and the volume of the rotating body can be solved by the definite integral. Therefore, definite integral has a wide range of applications in the field of science and engineering, such as circuits, mechanics, fluid mechanics, quantum mechanics. It also has application on other fields that will involve the operation of integral to promote the development of science, which makes it important for the whole science area. However, in the process of studying definite integrals, people will face many problems, such as insufficient theoretical support, lack of professional guidance, and insufficient timely summary in the research process. In order to solve these problems, researchers should correct their attitude towards scientific research, be realistic and down-to-earth. The purpose of this paper is to express the opinion on definite integral and hope to carry forward the great development of definite integral.

**Keywords:** Real analysis; Definite integral; Limit; Fourier transformation.

## 1. Introduction

Integration is a fundamental concept in calculus that focuses on finding the accumulated total of a given quantity over a specified interval. It is essentially the opposite operation of differentiation. The process of integration involves finding the anti-derivative (also known as the indefinite integral) of a function [1]. The anti-derivative of a function represents a family of functions, where each member of the family is a possible candidate for the original function when differentiated. The indefinite integral of a function is denoted by the symbol $\int$ and is followed by the function being integrated with respect to a variable of integration. The result is typically expressed as an equation with a constant of integration, symbolized by "C", since the anti-derivative is not unique and can have different solutions. The definite integral is an accumulated total of a function over a specific interval. It is denoted by $\int_a^b f(x)\,dx$ where a and b represent the upper and lower bounds of the integration, respectively. Integration has various applications in mathematics, physics, engineering, and other sciences. It is used to calculate areas, volumes, and rates of change, among other things. Integration is also utilized in solving differential equations, which describe how quantities change over time. Overall, integration plays a crucial role in mathematics and its applications, providing tools for analyzing and understanding the behavior of functions and their properties [2].

Newton and Leibniz independently discovered the central concepts of calculus, namely derivatives and integrals, at the end of the 17th century. However, due to the controversy and conflict between them, the contributions of the two scientists have also become the focus of controversy in history. Newton thought geometrically and applied the differential method to physics, thus establishing the basis of classical mechanics. Leibniz, on the other hand, paid more attention to semiotics and proposed the calculus algorithm, which laid the foundation for the development of mathematics and natural science. From a philosophical point of view, the definite integral can be seen as the understanding and description of continuity and change. It deals with the ever-changing phenomena of time, space and matter, as well as the concepts of infinite small quantities and limits. Imagine plotting a function on a graph, where the curve represents a specific phenomenon people are interested in [3]. By calculating the definite integral, one can determine the total area enclosed between the curve and the x-axis within a given range. This area serves as valuable information, as it represents a cumulative measure of change or accumulation within that interval. The introduction of definite integrals allows people to delve into the study of different variations in natural phenomena by examining the area beneath a function's curve. Moreover, this concept has broad applications when it comes to solving real-world problems and making predictions.

## 2. Theory of Integral

Integration is an essential process within calculus, serving as a fundamental tool for problem-solving in both mathematics and physics. In calculus people will learn about differentiation, integration, circular and hyperbolic functions and differential equations [4]. The Newton Leibniz formula, also known as the fundamental theorem, the fundamental theorem of calculus, or the first theorem of integration, is one of the important formulas in calculus. This formula is expressed as following. For a continuous function $f(x)$ at any point $x$ on the interval $[a,b]$, if the original function $F(x)$ of $f(x)$ exists, then [5]

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{1}$$

Among them, $\int$ represents the integral, $a$ and $b$ are the upper and lower bounds of the integral, $f(x)$ is the integrand function (also known as the integrand expression), and $F(x)$ is the original function of $f(x)$. The meaning of this formula is that for the integration of a continuous function on a certain interval, it can be obtained by finding the difference between the values of its original function at the endpoint of the interval.

There are some related corollaries of definite integrals, which are used for studying and calculating definite integrals [6].

The first is the additivity of integration. If function $f(x)$ is integrable on interval $[a,b]$, then for any $c$, $d$ satisfy $a \leq c \leq d \leq b$, and the integration of $f\ (x)$ on interval $[a,b]$ is equal to the sum of the integrals on interval $[a,c]$ and interval $[c,d]$, plus the integration on interval $[d,b]$, that is,

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_b^d f(x)dx \tag{2}$$

The second is the properties of integration. If functions $f(x)$ and $g(x)$ are integrable on intervals $[a,b]$ and $k$ is an arbitrary constant, then there are the following properties. One is that $\int_a^b kf(x)\,dx = k \int_a^b f\ (x)\,dx$, and the other is $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

The third is integral estimation. If the function $f\ (x)$ is non negative integrable on the interval $[a,b]$ and there are constants m and M, such that $m \leq f\ (x) \leq M$ holds for all $x$ on the interval $[a,b]$, then there is

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a). \tag{3}$$

The fourth is parity and Integration. If the function $f\ (x)$ is integrable on the interval $[-a,a]$ and satisfies one of the following conditions. If $f\ (x)$ is an odd function, then it is $\int_a^{-a} f(x)dx = 0$. If $f(x)$ is an even function, then $\int_{-a}^a f\ (x)\,dx = 2\int_0^a f\ (x)\,dx$

## 3. Applications

The first application is to prove the zeros of a given function. Let $f(x)$ be continuous on $(0,\pi)$, and let $\int_0^\pi f(x)\,cos\,k\,xdx = \int_0^\pi f(x)\,sin\,k\,xdx = 0$ for all $1 \leq k \leq n$. The purpose is to prove that $f(x)$ has at least $2n$ zeros on $(0,\pi)$ [7].

The author first observes that since $f(x)$ is continuous, it suffices to show that $f(x)$ changes sign at least 2n times on $(0, \pi)$. To do this, the paper considers $m$ distinct points $0 < x_1 < x_2 < \cdots < x_m < \pi, m \le 2n-1$, and the author wants to construct a function of the form

$$f(x) = \sum_{k=1}^{n} (ak \cos k\, x + bk \sin k\, x) \tag{4}$$

that changes sign in the vicinity of these points. For example, $f(x)$ is positive on $(0, x_1)$, negative on $(x_1, x_2)$, positive on $(x_2, x_3)$, negative on $(x_3, x_4)$, and so on. If $m < 2n-1$, the author can take $x_{m+1} = \cdots = x_{2n-1} = 0$.

Let $g(x) = \prod_{k=1}^{2n-1} \sin \frac{x-xk}{2}$, then the author has

$$g(x) = \prod_{k=1}^{2n-1} \left(\sin\frac{x}{2}\cos\frac{xk}{2} - \cos\frac{x}{2}\sin\frac{xk}{2}\right) = C \prod_{k=1}^{2n-1} \left(\sin\frac{x}{2} - ck\cos\frac{x}{2}\right). \tag{5}$$

Here, $C > 0$ and $c_1, c_2, \cdots, c_{2n-1} \le 0$. the author wants to prove that there exists $a \in [0, \pi]$ such that $f(x) = g(x) \cos\frac{x-a}{2}$. In this case, $f(x)$ satisfies the requirements. Using the sum and difference formula, the author can see that $g(x)\cos\frac{x-a}{2}$ is a trigonometric polynomial of degree $n$. So, the author only needs to prove that there exists $a \in [0, \pi]$ such that $\int_0^{2\pi} g(x) \cos\frac{x-a}{2} dx = 0$ This is equivalent to proving that

$$\int_0^{2\pi} g(x) \cos\frac{x}{2} dx = 2\int_0^{\pi} g(2x) \cos x\, dx \tag{6}$$

and

$$\int_0^{2\pi} g(x) \sin\frac{x}{2} dx = 2\int_0^{\pi} g(2x) \sin x\, dx \tag{7}$$

have opposite signs (considering one of them as zero also counts as having opposite signs). the author has

$$g(2x) = C \prod_{k=1}^{2n-1} (\sin x - ck \cos x) = \sum_{k=1}^{2n-1} (-1)^k ak \sin^{2n-1-k} \cos^k x \tag{8}$$

Here, $C > 0$ and $c_k$ are non-negative. From this, the author can immediately see that $\int_0^{\pi} g(2x) \sin x\, dx \ge 0$ and $\int_0^{\pi} g(2x) \cos x\, dx \le 0$. Therefore, the conclusion is established.

The second application is to calculate the limit that involves the integral, which is [8]

$$L = \lim_{n\to\infty} n^3 \left( \tan \int_0^{\pi} \sqrt[n]{\sin x}\, dx + \sin \int_0^{\pi} \sqrt[n]{\sin x}\, dx \right). \tag{9}$$

As $x$ approaches 0, it is easy to find that $\tan x - \sin x \sim \frac{x^3}{2}$. Therefore,

$$L = \lim_{n\to\infty} n^3 \left( \tan \int_0^{\pi} \left(\sqrt[n]{\sin x} - 1\right) dx - \sin \int_0^{\pi} \left(\sqrt[n]{\sin x} - 1\right) dx \right). \tag{10}$$

The limit above can be further simpified as

$$L = \lim_{n\to\infty} \frac{\left(n\int_0^{\pi}\left(\sqrt[n]{\sin x} - 1\right)dx\right)^3}{2} = \frac{\left(\int_0^{\pi} \ln\sin x\, dx\right)^3}{2} = -\frac{(\pi \ln 2)^3}{2}. \tag{11}$$

In the above calculation, the integral relation $\lim_{n\to\infty} n\int_0^{\pi}\left(\sqrt[n]{sin\,x}-1\right)dx = \lim_{n\to\infty}\int_0^{\pi}\frac{\sqrt[n]{sin\,x}-1}{\frac{1}{n}}dx = \int_0^{\pi}\ln(sin\,x)\,dx = -\pi\ln 2$ is used.

The third application is on the Fourier transform [9]. the author expands the function $f(x) = cos\,a\,x(a \notin Z)$ on the interval $[\pi, -\pi]$ into a Fourier series. By extending $f$ to a function with period $2\pi$ on the entire real number line, denoted as $\tilde{f}$, $\tilde{f}$ becomes a continuous even function with a period of $2\pi$ on the interval $(-\infty, +\infty)$. Therefore, since $a_n = \frac{2}{\pi}\int_0^{\pi} cos\,a\,x\,cos\,n\,xdx = \frac{1}{\pi}\int_0^{\pi}[cos(a-n)\,x + cos(a+n)\,x]dx$, it is found that

$$a_n = \frac{1}{\pi}\left[\frac{sin(a-n)\,\pi}{a-n} + \frac{sin(a+n)\,\pi}{a+n}\right] = \frac{(-1)^n}{\pi}\frac{2a\,sin\,a\,\pi}{a^2-n^2}(n = 0,1,2\cdots), \tag{12}$$

and $b_n = 0\ (n = 1,2,\cdots)$. According to Dini's theorem, the author obtains the Fourier series expansion of $\tilde{f}$ as

$$\tilde{f}(x) = \frac{sin\,a\,\pi}{\pi}\left[\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}cos\,n\,x\right]. \tag{13}$$

That is restricted to the interval $(-\infty, +\infty)$. Thus, on the interval $(-\infty, +\infty)$, the author has

$$cos\,a\,x = \frac{sin\,a\,\pi}{\pi}\left[\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}cos\,n\,x\right]. \tag{14}$$

If one takes $x = 0$ in Eq. (13), it is found that

$$\frac{\pi}{sin\,a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2} \quad (a \notin Z). \tag{15}$$

By contrast, $x = \pi$ is taken, it is arrived that

$$cos\,a\pi = \frac{sin\,a\,\pi}{\pi}\left[\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n\frac{2a}{a^2-n^2}\right]. \tag{16}$$

Before closing this part, the following relations hold by letting $a\pi = t$:

$$cos\,t = \frac{1}{t} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2t}{t^2-n^2\pi^2}, \qquad \frac{1}{sin\,t} = \frac{1}{t} + \sum_{n=1}^{\infty}(-1)^n\frac{2t}{t^2-n^2\pi^2}, \tag{17}$$

where $t \neq 0, \pm\pi, \pm 2\pi\cdots$.

The last application is on calculating the integral [10]

$$I = \int_0^{\infty}\left(\frac{a}{sinh\,a\,x} - \frac{b}{sinh\,b\,x}\right)\frac{dx}{x}. \tag{18}$$

First, let $t = iax$ in Eq. (15), the author has the relation

$$\frac{a}{sinh\,a\,x} = \frac{1}{x} + 2\sum_{n=1}^{\infty}(-1)^n\frac{ax}{a^2x^2+n^2\pi^2}. \tag{19}$$

Here, the author has used the formula $sinh\,x = \frac{sin(ix)}{i}$. Thus,

$$I_1 = \int_0^\infty \left(\frac{a}{sinh\, a\, x} - \frac{1}{x}\right)\frac{dx}{x} = 2\sum_{n=1}^{\infty}\int_0^\infty \frac{(-1)^n a^2 x}{a^2x^2 + n^2\pi^2}\, dx = 2\cdot\frac{\pi a}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -a\ln 2\,. \quad (20)$$

Similarly, it is easy to find that

$$I_2 = \int_0^\infty \left(\frac{b}{sinh\, b\, x} - \frac{1}{x}\right) dx = -\, b\ln 2\,. \quad (21)$$

Therefore, $\mathrm{I} = \int_0^\infty \left(\frac{a}{sinhax} - \frac{b}{sinhbx}\right) dx = I_1 - I_2 = (b-a)\ln 2$.

## 4. Conclusion

In conclusion, integration is a powerful tool in calculus that allows people to find the accumulated total of a quantity over a given interval. It is the opposite operation of differentiation and involves finding the antiderivative of a function. Through integration, the author can calculate areas, volumes, and rates of change, as well as solve problems related to differential equations. It has widespread applications in numerous fields, including mathematics, physics, engineering, and the sciences. Integration is denoted by the symbol $\int$ and is expressed as an equation with a constant of integration, representing the family of antiderivative functions. The definite integral provides the accumulated total over a specific interval. By utilizing integration, people gain a deeper understanding of the behavior and properties of functions. It is an essential concept that forms the foundation for further mathematical analysis and problem-solving. The examples in application illustrate the practical applications of definite integration in various fields such as physics, engineering, and economics, where the ability to calculate accumulated quantities, areas, distances, or averages is crucial for analysis and problem-solving.

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