Unravelling Three Differential Mean Value Theorems in Calculus

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Abstract. The theoretical foundation of differential aspect in real analysis is the differential mean value theorem, which connects to both the local and total properties of the copula function. The link between the copula function's local and overall properties is differential calculus. The maximum value, minimum value, extreme value, monotonicity, and other difficulties may all be resolved with the help of these differential mean value theorems. Additionally, they aid in the computation of limits, the demonstration of inequality, and the identification of the curve's inflection point and concave-convex interval. They are also capable of mapping the function and locating the equation's root. To tackle these issues, one can apply a number of significant findings from the differential mean value theorem. This study gives the formulae for solving three distinct mean value theorems. The Cauchy, Lagrange, and Roller theorems are examples of these mean value theorems. Understanding the connections between the three mean value theorems is made easier by these studies, which also provide an explanation of the theory underlying the theorems and provide examples and visuals of their practical application.

Keywords: Mean value theorem; Definite integral; Differential calculus.

1. Introduction

The mean value theorem (MVT) originated with the Indian mathematician Parameshvara in the 1300s and finally relied upon the scholarship of Augustin Louis Cauchy in 1823 and Michel Rolle in 1691. The formal presentation of these theorems and an example of the theorem represent a cornerstone of mathematics. The median theorem plays a key role in stating the fundamental theorem of calculus. F has a lower bound and upper bond on [a, b], if f is smooth on [a, b] and f' exists and is bounded on the interior, as is the case [1].

When calculus was initially invented in the 17th century by Gottfried Leibniz and Isaac Newton, it was a powerful instrument for studying mathematics. Most mathematicians have since then turned their attention to this potentially rewarding field of study. Scholars studying mathematics have been studying calculus ever since. Researchers were looking at something called the differential median theorem even before calculus was invented. The renowned French mathematician Fermat initially introduced the Fermat's theorem in his 1637 publication, "Methods of finding maximum and minimum values." The earliest polynomial version of Rolle's Median Theorem was published in the late seventeenth century by Rolle in his paper "Solutions of Equations." The first iteration of Lagrange's Median Theorem and its first proof were published in 1797 by the renowned French mathematician Lagrange in "Theory of Analytic Functions." However, the true meaning of the meticulous investigation of the differential median theorem may be attributed to the eminent French mathematician Cauchy. In his three excellent books, he offered the rigorization of the theory of mathematical analysis and rigorized the definition of the theory of calculus. Additionally, he tightened up the definition of mathematical analysis theory [2]. He started by going into more detail about the differential median theorem's important purpose, which he positioned as the core idea of differential calculus.

Differential calculus revolves with three different MVTs, which are of Rolle's type, Cauchy's type, and Lagrange's type. The Lagrange's MVT is very outstanding [3]. Because it establishes a quantifiable connection between the value and the derivative of a given function. The derivative of the MVT can be used for analyzing functions [4]. This is in order to potentially establish a measurable correlation between the function values and the derivative value. Derivatives can be found in the monotonicity of a function, extreme values, concave points, convex points, inflection points, and other important function states to understand a wide variety of functions. In differential calculus, the MVT is essentially a connecting line between two positions.

2. Rolle's Mean Value Theorem

2.1. Statements

Differential calculus provides a specific example of Rolle's theorem by analyzing these functions, which is called the MVT. According to Rolle's theorem, assuming the function "f" is smooth on the closed region $[\alpha, \beta]$, it has derivative on the region (α, β). Such that if f(α)=f(β) then $f'(x)$ will equal 0 for x which has the domain of $\alpha \leq x \leq \beta$ [5].

If a smooth curvture crosses the same value twice. Stated otherwise, the curve has to have a distinct tangent line also referred to as a derivative at every interval point. The first known statement of the theorem dates back to the 12th century and is attributed to the Indian scholar Bhaskara II. However, the French mathematician Michel Rolle did not present a rigorous modern proof of the thesis until 1691. Rolle's theorem is not particularly helpful outside the scope of proving the MVT.

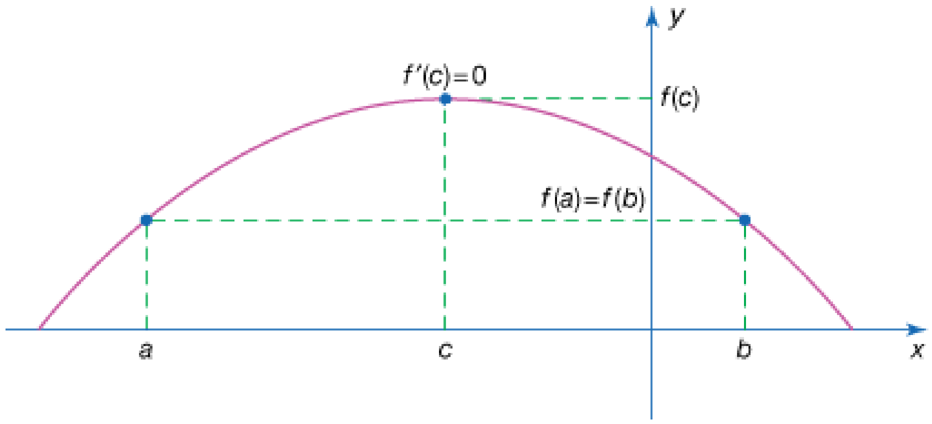

Fig 1. Illustration of Roller's MVT. The derivative of function at c is zero.

Besides affirming the coordination of time and space, Berlinski (1995) pointed out that Rolle's Theorem is about functions and, as such, a theorem involving processes represented by functions. The constraints deal with continuity and differentiability, two essential properties of mathematics. 1911–1922. Moreover, Rolle's Theorem demonstrates a connection between differentiability and continuity, according to Berlinski. A connection exists between reality and Roller's theorem. A ball that has been hurled high descends and, during its descent, alters course before landing. Therefore, the ball's thrown-upward velocity must finally drop to zero, as explained by Rolle's theorem.

Theorem 1. If f is a function that is derivative on (α, β) and smooth on $[\alpha, \beta]$, with $f(\alpha) = f(\beta) = 0$, then there exists some c in (α, β) where $f'(c) = 0$, see Fig. 1 [6].

Proof. Starts with the following two scenarios to illustrate the theorem. For every t in [α, β], f(x) = 0 in instance 1. Since the function takes a constant value on the region [α, β], and f'(x) = 0 when x is a constant, c in this case satisfies the conditions for the theorem to hold such that it takes any value between α and β. In example 2, f(x) does not equal to 0 for some argument in (α,β). Such value's absolute maximum and lowest are both contained in the closed shell by the extreme value theorem. Remember that f(α) = f(β) = 0 and that, in accordance with the hypothesis, f(x) is not zero in this case for some in (α, β). As a result, at some Cmax in (α, β). At some Cmin in (α, β), only two cases will occur: absolute maximum positive and absolute minimum negative or both. C should be set to either Cmin or Cmax depending on which you have. Thus, for any t in (α, β), the region (α, β) contains c and one of the following: f(c) > f(x); or f(c) < f(t). This suggests that in both scenarios, and c is a local extreme of f. Since f is similarly derivative at c and implies that f'(c) = 0, Fermat's Theorem is relevant.

2.2. Examples

The Fig. 2 is the figure of f(x) = sin(x) + 2 which owns a domain 0 < x ≤ 2π. "f" is smooth on [0,2π] and derivative on (0,2π), as shown by f(0) = f(2π) = 2 [7]. Thus, there are at least one value of x that equals c exists so f ′(c) = 0, in accordance with Rolle's theorem. Given that f'(x) = cos(x), and f ′(c) = cos(c) = 0 follows. On the interval [0, 2π], the above equation has two solutions: $c_1 = \pi/2$ and $c_2 = 3\pi/2$. As a result, as seen in Fig. 2 below, at $x = \pi/2$ and $x = 3\pi/2$, the graph has tangents with slopes of 0 (horizontal lines).

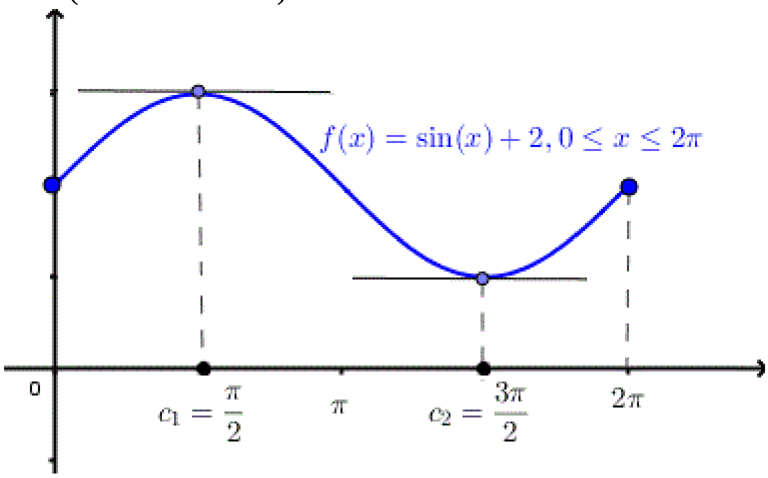

Fig. 2 The graph of function $f(x)$. It has maximum at c_1 and minimum at c_2.

3. Cauchy's Mean Value Theorem

3.1. Statements

One of the key theorems in differential calculus is known as Cauchy's MVT. It is based on differentiable and continuous functions. It explained the relationship between the average slope of the function and its instantaneous slope during a certain period of time. Cauchy, who dubbed his theory after himself, expanded Lagrange's MVT. It makes the function f(x) differentiable in the intervals (α, β) and α < β, and it makes it smooth on the confined region [α, β], the point ξ that is connected to (α, β) afterwards, such that [8]

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\xi)}{g'(\xi)}. \tag{1}$$

Cauchy's MVT may be understood geometrically as follows: a parametric equation with a tangential cord that parallels to that segment containing the two endpoints can represent at least one point on a curve. The statement "at least one point on a curve is represented by a parametric equation" refers to this. This theorem may be regarded of as the expression of Lagrange's MVT in parametric equations. The Fig. 3 illustrates the geometric meaning of Cauchy's MVT.

Concerning every variable x ∈ (α, β), derivative of g(x) is nonzero. If both f(x) and g(x) meet the requirements of being smooth on the confined region [α, β] and derivative in the region (α, β). After that, Eq. (1) is valid for at least one dot ξ in (α, β). The following may be produced based on the findings of Cauchy's MVT: $f'(\xi) - \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} g'(\xi) = 0$. If constructing an auxiliary function [9]

$$\varphi(x) = f(x) - \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} g(\xi). \tag{2}$$

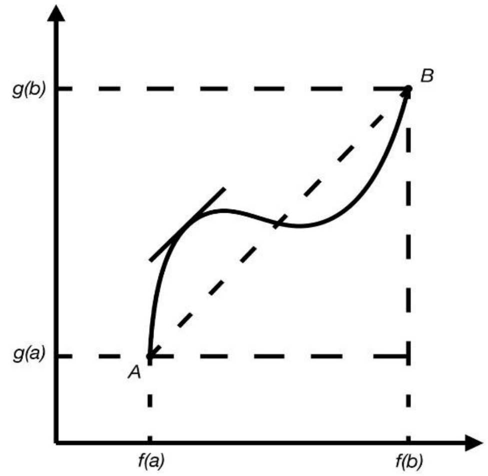

Fig 3. Illustration of Cauchy's MVT. Derivative at certain point is parallel to that of A to B.

Therefore, it must validate the equation. $\varphi'(\xi) = f'(\xi) - \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} g'(\xi) = 0$ to be true. This has to do with verifying Rolle's theorem; therefore, it must determine if φ(α) and φ(β) are equivalent. After some deduction, it is ultimately found that

$$\varphi'(\xi) = f'(\xi) - \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} g'(\xi) = 0. \tag{3}$$

After final arrangement it will be equal to Eq. (1).

3.2. Examples

If $f(x)$ is smooth at $[\alpha, \beta]$ and its derivative exists at $(\alpha, \beta)(\alpha > 0)$, it can prove that there is one dot $\xi \in (\alpha, \beta)$, such that $\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = \frac{\xi^2 f'(\xi)}{\alpha\beta}$. By dividing both sides by ab, then it follows that $\frac{f(\beta)-f(\alpha)}{1/\alpha-1/\beta} = \frac{f'(\xi)}{1/\xi^2}$, which can be recast as

$$\frac{f(\beta)-f(\alpha)}{1/\beta-1/\alpha} = \frac{f'(\xi)}{-1/\xi^2}. \tag{4}$$

If one sets $g(x) = \frac{1}{x}$, it is found that $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\xi)}{g'(\xi)}$ by Cauchy's theorem, then Eq. (4) holds.

4. Lagrange's Mean Value Theorem

4.1. Statements

Lagrange's mean value theorem is the most important one among several mean value theorems. It is the bridge of differential calculus application and plays an important role in some theoretical derivation of higher mathematics, and has extremely high research value in theory and practice. It is stated as [10]

$$f'(c) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}. \tag{5}$$

The following Fig. 4 provides an illustration of Lagrange's MVT. Understanding Lagrange's MVT can be done from multiple angles, drawing and observing that the equation $y = f(x)$ is a proof of geometric perspective. The authors first take two dots, (α, f(α)) and (β, f(β)), and the curved line $y = f(x)$ passes through these two points, and the dots (c, f(c)) are the midpoint of the curve between these two points. After this (α, f (α) and (β, f (β)) of the tangent of the slope (f (β) - f (α))/(β - α). f'(c) is the obliquity of the tangent line intersecting the curvature at the dot (c, f(c)). Thus, according to Lagrange's MVT, the tangential cords going through the dots (α, f(α)) and (β, f(β)) are exactly parallel with tangential lines of the points (c, f(c)), which they have the same slope. Therefore, f'(c) = (f(β) − f(α))/(β − α) will be its result.

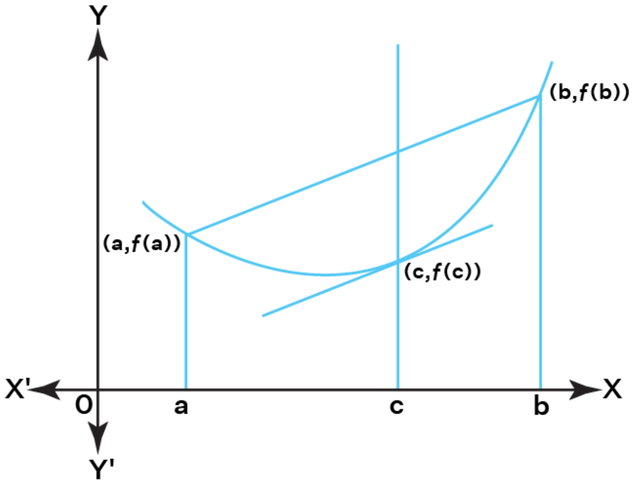

Fig 4. Illustration of Lagrange's MVT.

If the constructor $f(x)$ meets the requirements to have derivative on the open interval (α, β) and be smooth on the confined region [α, β]. So, there is at least some $\xi(\alpha < \xi < \beta)$ that meets

$$F(\beta)\text{–}f(\alpha) = f'(\xi)(\beta - \alpha). \quad (6)$$

The authors propose an auxiliary function $\Phi(x) = f(x) - f(\alpha) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}(x - \alpha)$ and show that $\Phi(x)$ is smooth on [α, β] to support this claim. In (α, β), it has derivative and $\Phi(\alpha) = \Phi(\beta)$. The Rolle's theorem states that $\phi'(\xi) = 0$ for $\exists\xi \in (\alpha, \beta)$. However,

$$\phi'(x) = f'(x)\text{–}\frac{f(\beta) - f(\alpha)}{\beta - \alpha}. \quad (7)$$

Consequently, $\phi'(\xi) = f'(\xi) - (f(\beta) - f(\alpha))/(\beta - \alpha) = 0$. So, $(f(\beta) - f(\alpha))/(\beta - \alpha) = f'(\xi)$. Thus, $f'(\xi)(\beta - \alpha) = f(\beta) - f(\alpha)$.

4.2. Statements

The following areas have seen extensive use of the Lagrange MVT. It is capable of proving inequality, proving equations, and studying the characteristics of functions and derivatives. For the identityarcsin x + arccos $x = \frac{\pi}{2}(-1 \le x \le 1)$, it is assumed that $F(x) = \arcsin x + \arccos x - \frac{\pi}{2} when F(0) = 0$. It is evident that $F(x)$ is derivative in (− 1,1) and smooth on [− 1,1]. Thus, $F(x) - F(0) = F'(\xi)$ may be reasoned from the Lagrange MVT $\exists\xi \in (-1, 1)$. However, $F'(x) = \frac{1}{\sqrt{1-x^2}}\text{-}\frac{1}{\sqrt{1-x^2}} = 0$. $F(x) - F(0) = 0$ as a result of $F'(\xi) = 0$. Specifically, $F(x) = F(0) = 0$, then $F(x) = \arcsin x + \arccos x - \frac{\pi}{2} = 0$. Therefore, $\arcsin x + \arccos x = \frac{\pi}{2}$.

In addition, it can prove differential inequality. When $t > 0, t/(t + 1) < ln(t + 1) < t$. Let $f(p) = lnp$, then $f'(p) = \frac{1}{p}$,$f(1) = ln1 = 0$. Obviously, $f(p)$ is in line with the Lagrange MVT $[1,1 + t](t > 0)$, so there is$\xi \in (1,1 + t)$ such that

$$f(1 + t) - f(1) = f'(\xi)[(1 + t) - 1] \quad (8)$$

Namely, $\ln(1 + x) - \ln 1 = \frac{x}{\xi}(1 < \xi < 1 + x)$. Because $1 < \xi < 1 + t$, so $1/(1 + t) < 1/\xi < 1$. Therefore, $t/(1 + t) < t/\xi < t$ and thus $t/(t + 1) < \ln(t + 1) < t$.

5. Conclusion

The link between the three MVTs can be further clarified as follows after the full-text explanation and illustration. The three MVTs are based on the evaluation concept of derivatives, which determine that if a function exists up to a particular value within a given range. The three MVTs examine a function's properties throughout a certain period, which may be utilized to solve various problems, generate inequalities, and determine a function's maximum value. Similar techniques may be used to prove each of the three theorems. They all embrace the intermediate value theorem and extend the function using Taylor's formula. The uniqueness and differentiation of the three MVTs can also be proved by the following methods. The constructor must be derivative only in the confined region in order to comply with Lagrange and Rolle's MVT. More stringent requirements, such as a function that must be smooth in a confined region or derivative in an open region, are related to Cauchy's MVT. Unlike Rolle's MVT, which requires a function to take the same value at both ends of an interval, Lagrange's and Cauchy's MVT seek to find a point in an interval where the derivative of a constructor is equal to a certain value.

Authors Contribution

All the authors contributed equally and their names were listed in alphabetical order.

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