Investigate the Theoretical Significance and Applications for Differential Geometry

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Abstract. With the development of modern science and technology, mathematics, physics, and fluid science have played a crucial role in scientific progress. As an important branch of mathematics, differential geometry is the key to studying these disciplines, which have a wide range of applications in fields such as mathematics and applied science. At present, mathematicians have not made ideal progress in the field of differential geometry research. As one of the important research topics in the current mathematical community, differential geometry research also has extensive research value and can provide mathematical models for the development of other scientific and technological fields. Therefore, this study mainly focuses on the development history and some basic knowledge of differential geometry. The application of differential geometry is explored through the study of the origin and development of differential geometry. This paper aims to find that differential geometry has essential research value and is indispensable in developing basic disciplines and future science.

Keywords: Differential geometry; theoretical introduction; applications.

1. Introduction

Differential geometry is a branch of science that studies geometric structures such as curves, surfaces, and manifolds. The origin of differential geometry can be traced back to countless mathematicians who have successively devoted themselves to the study of differential geometry, making important contributions to human academic research. Differential geometry also plays a crucial role in many disciplines, from the 19th century, when Gauss and Riemann elevated the subject of geometry to an entirely new field, to contemporary differential geometry: differential geometry further developed in the latter half of the 20th century and the 21st century, involving more abstract and complex concepts and methods. Modern differential geometry is a powerful tool for the study of general relativity. Most of the monographs and textbooks of general relativity published by international frontier scholars (such as Hawking) use the overall differential geometry language. More and more domestic physicist workers (especially young and middle-aged) hope to learn this language. However, due to differential geometry's extensive and profound nature, a capital difficult word has become a common obstacle beginners encounter. Most people hesitate, want to enter no door, or taste [1]. Researchers have conducted more in-depth research in Riemannian manifolds, Riemannian metrics, curvature, metrics, submanifolds, and more. In addition, differential geometry has extensive cross-applications with fields such as mathematical physics, topology, and geometric analysis.

This paper aims to introduce the development history and basic knowledge of differential geometry, attract more mathematical enthusiasts' interest in differential geometry, and promote the development of differential geometry research.

2. Origin and Development of Differential Geometry

Differential geometry, partial differential equations and mathematical physics are the three most active branches of core mathematics in the world today [2]. The origin of differential geometry can be traced back to Gauss and Riemann in the 19th century. Gauss studied curves and surfaces and proposed Gaussian curvature: (Gaussian curvature is an intrinsic measure of curvature: usually represented by K, and at any point on the surface, there are principal curvatures K_1, K_2, and Gaussian curvature $K = K_1 * K_2$, which can also be expressed as ϕ Gauss curvature K of $\phi :=$

$Det\ \omega\phi$, Another knowledge point can be obtained from it: the umbilical point. The principal curvatures K_1, K_2. If $K_1(u) = K_2(u)$ holds, it calls the navel point of $u \in D\ \phi$). Riemann extended the concept of geometry to high-dimensional spaces and created Riemannian geometry, extending the study of differential geometry from planar curves and surfaces to more general manifolds. The hypothesis of geometric foundations can be said to be the initial origin of differential geometry [3].

In the first half of the 20th century, differential geometry was widely applied in physics. Einstein's general theory of relativity combined differential geometry with the theory of gravity, driving the development of differential geometry and proposing Einstein's differential equation: $R\mu\nu - 1/2Rg\mu\nu = 8\pi T\mu\nu$. Among them, $R\mu\nu$ is the curvature tensor of spacetime, R is the curvature scalar, $g\mu\nu$ it is a metric tensor of spacetime, $T\mu\nu$ it is a tensor of matter and energy. This equation means that the curvature of spacetime is determined by matter and energy. If there were no matter and energy, spacetime would be flat, so it can obtain some basic knowledge.

A regularized parameterized curve in R^2 refers to a curve that satisfies: Smooth mapping with $dt = 0(t \in I)\ \gamma : I \to R^2$, where $I \subseteq R$ is an interval for $\gamma = (x_1, x_2)^T$ Let's note $d\gamma/Dt := (\frac{dx_1}{dt}, \frac{dx_2}{dt})^T$. It calls $t \in I$ a regular parameterized curve γ: Parameters for $I \to R^2$.

A parameterized curve with arc length in R^2 refers to a curve that satisfies: $|((d\gamma)/ds)| = 1$. A smooth mapping on $I \to R^2$, and let $s \in I$ be γ: Arc length parameter for $I \to R^2$.

Bertrand Curve and Mannheim curve are two kinds of classical curves, which are described in many differential geometry textbooks [4].

Next, it can introduce curve theory: for any given interval $0 \in I \subseteq R,\ f \in C\infty(I)$, and $R^2{}'s$ (forward) unit orthogonal basis $[v_1, v_2] \in SO\ (2)$, there exists a unique arc length parameterized curve $\gamma : I \to R^2$ meets:

$K_r = f,\ \gamma(0) = \gamma_0, \gamma$. Frenet frame $F\ (0)\ at\ s = 0 = [v_1, v_2]$.

The Frenet frame is a commonly used coordinate system in differential geometry and curve theory, consisting of three vectors: tangent vector, average vector, and binormal vector. The advantage of Frenet frames is that they can provide local properties of curves at each point, such as curvature. Frenet frames are widely used in computer graphics, robotics, and computer-aided design fields. It can be used for tasks such as path planning, motion planning, and surface reconstruction, as well as for describing and analyzing complex geometric shapes.

Next, analyze curvature and torsion: assuming that the general parameter of the curve (c) on R^3 is expressed as $r = r\ (t)$.

The formula for calculating its curvature and torsion is [5]:

$$\kappa = \frac{|r' \times r''|}{|r'|^3}, \tag{1}$$

$$\tau = \frac{(r', r'', r''')}{|r' \times r''|}. \tag{2}$$

On the basis of the curve, the spatial dimension is increased to obtain the surface of differential geometry, which mainly has the following two forms

Let $D \subseteq R^2$ be the open area, $\phi : D \to R^3$ is a smooth mapping for $u = (u_1, u_2) \in D$, remember, $\phi(u) = (x_1(u), x_2(u), x_3(u))^T$, $\phi_i = (\partial_{u^i}x_1, \partial_{u^i}x_2, \partial_{u^i}\ x_3)$, $i = 1{,}2\ If\ for\ any\ u \in D$, $\phi_1(u)\ and\ \phi_2(u)$ linearly independent, then $\phi : D \to R^3$ is a regularized parameterized surface.

If $|\phi_1| = |\phi_2|, \phi_1 \cdot \phi_2 = 0.\ holds\ on\ D$, then it is called $\phi : D \to R^3$ is conformal parameterization.

A regular parameter curve in R^3 is a smooth map $\gamma: I \to R^3$ that satisfies $d\gamma/dt = 0(t \in I)$, where $I \in R$ is an interval. It calls $t \in I$ the parameter of the regular parametric curve $\gamma: I \to R^3$. Let $\gamma: I \to R^3$ is a curve of arc length, known from $d\gamma/ds \cdot d\gamma/ds = 1$.

$$\frac{d\gamma}{ds}\cdot\frac{d^2\gamma}{ds^2}=0(s\in D)\,. \tag{3}$$

That is, $\frac{d\gamma}{ds}$ and $\frac{d^2\gamma}{ds^2}$ are orthogonal everywhere. Thus, a set of bases of R^3 can be introduced at $\frac{d^2\gamma}{ds^2}\neq 0$.

$$T(s):=\frac{d\gamma}{ds},N(s):=\frac{\frac{d^2\gamma}{ds^2}}{\left|\frac{d^2\gamma}{ds^2}\right|},B(s):=T(s)\times N(s). \tag{4}$$

Thereby,

$$F'(s)=[T(s),N(s),B(s)]\in SO(3),s\in I\,. \tag{5}$$

Since$F(S)^{-1}$, $dF/(ds)(s)$ is an argument $3*3$ matrix, namely the presence of $\lambda,\tau,a\in C^{\infty}(I)$.

$$F^{-1}\frac{dF}{ds}=\begin{bmatrix}0&\kappa&\lambda\\-\kappa&0&\tau\\-\lambda&-\tau&0\end{bmatrix}. \tag{6}$$

From the definition of N, it is easy to prove $\lambda=0$, it calls κ and τ is the curvature and lection of γ, respectively.

In the latter half of the 20th century, differential geometry was further developed and generalized. Gregory Perelman's contributions to manifold topology and geometry were awarded the Fields Prize, and their work provided essential tools and methods for understanding the topological properties and geometric structures of manifolds.

In Perelman's work, he applied Grisha Perelman's pioneering method to establish proofs of the Poincar é conjecture and Riemann conjecture. He summarized his in-depth research on the geometric topology of manifolds, utilizing the theories of Ricky manifolds and Riemannian manifolds. Through his work, Perelman solved some fundamental mathematical problems and brought new insights to the entire field of differential geometry.

So far, differential geometry is still an active field in mathematics and physics, involving applications and research in multiple fields.

Mechanical ideas drive the development of differential geometry. When some mechanical concepts were produced, people hoped to explain the geometry theory, such as space-time manifolds or metric on spatial manifolds and geodesics. With the advent of more and more concepts, geometric ideas are also broadened, thus promoting the development of differential geometry [6].

3. Applications of Differential Geometry

Differential geometry has extensive applications in many fields, including physics, engineering, and computer graphics.

(1). Differential geometry can be applied to the spatial modeling of steel wire ropes, deriving the rotation angle of the strand in the wire rope, the curvature of any point on the strand, the torsion of any point on the strand, the internal standard frame of any point in the wire rope, and the calculation formula for the circumference of the steel wire in the strand in the wire rope, thus solving the problem of steel wire ropes [7].

(2). Differential geometry is used in computer graphics to model and render surfaces, enabling computers to generate realistic three-dimensional shapes. It can describe and calculate the average vector, curvature, texture coordinates, and other surface attributes, thereby achieving lighting and texture mapping effects.

(3). Differential geometry is used in geography and geophysics to model and analyze phenomena such as topography, crustal movement, and geological structure on the Earth's surface. With the help of differential geometry tools, it is possible to calculate the curvature, average vector, and area attributes of a surface, thereby deriving the shape and geographic information of the Earth.

(4). In medical image analysis and high-level computer vision, there is intensive use of geometric features like orientations, lines, and geometric transformations ranging from simple ones (orientations, lines, rigid body or affine transformations, etc.) to very complex ones like curves, surfaces, or general diffeomorphic transformations [8].

(5). By applying the concept of differential geometry, architects can create structures with specific curvature, surface features, and geometry. This can lead to more attractive and unique architectural designs. Due to increasing urbanization and rapid infrastructural development, scarcity of land is a problem [9]. So the buildings where people live are more three-dimensional and beautiful

(6). Differential geometry is also widely used in the field of materials, dislocation-based modeling, and numerical analysis for kink deformation using the theory of nonlinear continuum mechanics based on differential geometry to discuss the material strengthening mechanism [10].

4. Conclusion

This paper mainly introduces the development history and basic introductory knowledge of differential geometry, including surfaces, curves, frenet frames, curvature, and other basic knowledge. In the university undergraduate mathematics course, only differential geometry is not a compulsory course, which also indirectly reflects the research difficulty of differential geometry in the field of mathematics The study of differential geometry poses certain challenges in thoroughly studying this field. Differential geometry plays an important role in many disciplines and application fields, helping us understand and describe the characteristics of curves and surfaces, and providing mathematical tools and methods for modeling and solving practical problems. The development of artificial intelligence, as well as basic physics, require differential geometry theory as theoretical support.

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