Weierstrass Representation of Minimal Surfaces and Related Basic Quantities

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Abstract. Minimal surface, as an important surface in differential geometry, has long been one of the research topics of many scholars. It provides far-reaching research materials for geometric analysis and nonlinear partial differential equations, and plays an important role in mathematical general relativity. The minimal surface mentioned in this paper refers to the surface with the smallest area when the boundary conditions remain unchanged, i.e., the Plateau problem. The physical counterpart is the soap film experiment. It is different from a surface of constant mean curvature in another sense. Weierstrass discovered that the general solution of minimal surface equations can be given by complex analysis, that is, Weierstrass representation of minimal surface, thus revealing the essential relationship between minimal surface and holomorphic function and meromorphic function. In this paper, the Weierstrass representation of minimal surface is organized, and the first and second basic forms of minimal surface are derived by using its complex vector form. The study of minimal curved surface has played an important role in many fields such as construction engineering, material science and so on.

Keywords: Minimal surface; conformal parameterized surface; Weierstrass representation; the first and second basic forms of minimal surfaces.

1. Introduction

Minimal surface, as an important surface in differential geometry, has long been one of the research topics of many scholars, and is widely used in many fields such as construction engineering and material science. By the 19th century Belgian physicist J. Plateau's soap film experiment, it was learned that a metal wire is bent into a closed space curve, and it is immersed in soap solution, and after taking it out, the film with the smallest surface area of the metal wire boundary can be seen on the surface of the metal wire due to the surface tension. Nowadays, the problem of finding a surface with the smallest area bounded by a given space curve C is usually called the Plateau problem [1]. In mathematics, the film with the smallest surface area is called a minimal surface. Euler had proposed a similar problem of minimal surfaces as early as 1744, but it is generally believed that Lagrange was the first to analyze and discuss the problem of minimal surfaces. He used variational method to transform a nonlinear optimization problem into a partial differential equation problem, and obtained a second order elliptic partial differential equation. Later, in 1776, the geometer J. B. Merusnier gave a geometric explanation of the equation of a minimal surface, that is, the average curvature of every point on the surface is 0 [2].

Minimal surfaces exist widely in nature and are important surfaces whose mean curvature is 0 everywhere. It has a rich and compatible mathematical structure, which can be dealt with by different mathematical viewpoints and methods. The locally minimal surface is considered to be defined as a critical point of the area generalized function, denoted by the variational equation or the equivalent Euler-Lagrange equation (second-order elliptic partial differential equations). Minimal mapping is a special mapping between two Riemann manifolds. Weierstrass abandoned the concept of area and gave the general solution of minimal surface equation from the perspective of complex analysis [3].

In 2021, Davor and Željka gave a derivation of a Weierstrass representation of regular light-like surfaces in Lorentz-Minkowski three-dimensional space. For regular light-like surfaces, they are locally parameterized with a pair of holomorphic and meromorphic dual functions. Moreover, the definition of minimal light-like surfaces in Lorentz-Minkowski three-dimensional space is also discussed on this basis, and it is related to the class of surfaces characterized by the Weierstrass representation formula [4]. In addition to this, in another paper by Davor, he extended the formula for the Weierstrass representation of the minimal class of time-like surfaces in Minkowski three-dimensional space found by S. Lee to the same surfaces in Minkowski four-dimensional space [5].

The basis of complex analysis methods for minimal surfaces is the Weierstrass representation, with which the function theory method has become an important tool for studying minimal surfaces [6].

Minimal mapping is a special mapping between two Riemann manifolds. The Weierstrass representation discussed in this paper abandons the concept of area and discusses minimal surface from the point of view of special mapping. Through the concepts and methods of complex functions such as isothermal coordinates and analytic functions, the general solution of minimal surface equation is given by complex analysis [7].

This paper first discusses the Weierstrass representation of minimal surfaces and its applications, and then gives some basic geometric quantities as well as the first and second basic forms of minimal surfaces by using the holomorphic and meromorphic functions in its formulation.

2. Basic knowledge preparation

2.1. Regular Parameterized Surface

Definition 2.1.1 [6]. Let $\mathrm{D} \subseteq \mathbb{R}^2$ be an open region and $X: D \to \mathbb{R}^3$ be a smooth mapping. For $(u, v) \in D$, denote by:

$$\mathrm{X}(\mathrm{u},\mathrm{v}) = \left(x_1(\mathrm{u},\mathrm{v}), x_2(\mathrm{u},\mathrm{v}), x_3(\mathrm{u},\mathrm{v})\right), \quad \frac{\partial \mathrm{X}}{\partial \mathrm{u}} = \left(\frac{\partial x_1}{\partial \mathrm{u}}, \frac{\partial x_2}{\partial \mathrm{u}}, \frac{\partial x_3}{\partial \mathrm{u}}\right), \frac{\partial \mathrm{X}}{\partial \mathrm{v}} = \left(\frac{\partial x_1}{\partial \mathrm{v}}, \frac{\partial x_2}{\partial \mathrm{v}}, \frac{\partial x_3}{\partial \mathrm{v}}\right). \tag{1}$$

If $\frac{\partial \mathrm{X}}{\partial \mathrm{u}}$ and $\frac{\partial \mathrm{X}}{\partial \mathrm{v}}$ are linearly independent for any $(u, v) \in D$, then $X: D \to \mathbb{R}^3$ is said to be a regular parametric surface.

Given a smooth simple closed curve in $\mathbb{R}^3$, the surface with the smallest area is called minimal surface, and the Euler-Lagrange equation of area variational is called minimal surface equation [8].

2.2. Minimal Surface and Minimal Surface Equation

The definition of minimal surface is given first.

Definition2.2.1 [9]. In Euclidean space, when the mean curvature of a regular parameterized surface is constant zero (i.e. $\mathrm{H} \equiv 0$), the surface is said to be a minimal surface, i.e., a surface whose area reaches a critical value under normal variation. In common parlance, it is the surface whose area is minimized by the boundary of a given space curve.

Then the minimal surface equation is derived from the definition of the graph of the function.

Definition2.2.2 [6]. Let D be an open region in the plane, $f \in C^\infty(D)$ and $X: D \to \mathbb{R}^3$, $(u, v) \longmapsto (u, v, f(u, v))$.

A regular parameterized surface is defined, and the surface is said to be the graph of a function f.

A direct computation yields:

$$\frac{\partial X}{\partial u} = (1,0, f_u), \frac{\partial X}{\partial v} = (0,1, f_v). \tag{2}$$

The normal vector is expressed as:

$$\mathrm{n} = \frac{(-f_u, -f_v, 1)}{\sqrt{1+f_u^2+f_v^2}}. \tag{3}$$

The first basic quantities are:

$$g_{11} = 1 + f_u^2,\ g_{12} = g_{21} = f_u f_v,\ g_{22} = 1 + f_v^2. \tag{4}$$

The second basic quantities are:

$$h_{11} = \frac{f_{uu}}{\sqrt{1+f_u^2+f_v^2}}, h_{12} = h_{21} = \frac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}}, \quad h_{11} = \frac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}}. \tag{5}$$

The mean curvature is expressed as:

$$\mathrm{H} = \frac{(1+f_u^2)f_{vv}+(1+f_v^2)f_{uu}-2f_uf_vf_{uv}}{2(1+f_u^2+f_v^2)^{\frac{3}{2}}}. \tag{6}$$

The Gauss curvature is given by:

$$\mathrm{K} = \frac{f_{uu}f_{vv}-{f_{uv}}^2}{(1+f_u^2+f_v^2)^2}. \tag{7}$$

From the definition of minimal surface, it knows that when H=0, it has:

$$(1+f_u^2)f_{vv} + (1+f_v^2)f_{uu} - 2f_uf_vf_{uv} = 0. \tag{8}$$

This equation is known as the equation of the minimal surface [6].

2.3. Conformal Parameterized Surface

Definition 2.3.1 [6]. If $\left|\frac{\partial \mathrm{X}}{\partial \mathrm{u}}\right| = \left|\frac{\partial \mathrm{X}}{\partial \mathrm{v}}\right|$, and $\frac{\partial \mathrm{X}}{\partial \mathrm{u}} \cdot \frac{\partial \mathrm{X}}{\partial \mathrm{v}} = 0$ holds on D, then $\mathrm{X}: \mathrm{D} \to \mathbb{R}^3$ is said to be a conformally parametrized surface, i.e., an orthogonal parametric surface that satisfies $g_{11} = g_{22}$.

2.4. The First Basic Form of a Surface

There is a one-to-one correspondence between the conformal class of the first basic form of a surface and its complex structure.

Definition 2.4.1 [10]. If the equation for surface S is $\mathrm{r} = \mathrm{r}(\mathrm{u}, \mathrm{v})$, then it has $\mathrm{ds}^2 = \mathrm{dr}^2 = (\mathrm{r}_u du + \mathrm{r}_v dv)^2 = {\mathrm{r}_u}^2\,\mathrm{du}^2 + 2\mathrm{r}_u\mathrm{r}_v dudv + {\mathrm{r}_v}^2\,\mathrm{dv}^2$, which is called the first basic form of a surface. Denoted as:

$$\mathrm{I} = g_{11}\mathrm{du}^2 + 2g_{12}dudv + g_{22}\,\mathrm{dv}^2. \tag{9}$$

Among them, $g_{11} = \mathrm{r}_u \cdot \mathrm{r}_u$, $g_{12} = \mathrm{r}_u \cdot \mathrm{r}_v$, $g_{22} = \mathrm{r}_v \cdot \mathrm{r}_v$ is called the first basic quantity of surface S.

The unit normal vector is represented as:

$$\mathrm{n} = \frac{(\mathrm{r}_u \times \mathrm{r}_v)}{|\mathrm{r}_u \times \mathrm{r}_v|}. \tag{10}$$

When $\mathrm{X}(\mathrm{u}, \mathrm{v}): \mathrm{D} \to \mathbb{R}^3$ is a conformal parameterized minimal surface, it has $g_{11} = g_{22}$, $g_{12} = 0$, then the first basic form of $\mathrm{X}(\mathrm{u}, \mathrm{v})$ is:

$$\mathrm{I} = g_{11}(du^2 + dv^2) = g_{11}|dz|^2. \tag{11}$$

2.5. The Second Basic Form of a Surface

Definition 2.5.1 [10]. If the equation for surface S is $\mathrm{r} = \mathrm{r}(\mathrm{u}, \mathrm{v})$ and the unit normal vector is n, then it has $\mathrm{n} \cdot \mathrm{dr}^2 = \mathrm{n} \cdot \mathrm{r}_{\mathrm{uu}}\,\mathrm{du}^2 + 2\mathrm{n} \cdot \mathrm{r}_{\mathrm{uv}}\mathrm{dudv} + \mathrm{n} \cdot \mathrm{r}_{\mathrm{vv}}\,\mathrm{dv}^2$,which is called the second basic form of a surface. Denoted as:

$$\mathrm{II} = \mathrm{n} \cdot \mathrm{dr}^2 = h_{11}\mathrm{du}^2 + 2h_{12}dudv + h_{22}\,\mathrm{dv}^2. \tag{12}$$

Among them, $h_{11} = \mathrm{r}_{uu} \cdot n$, $h_{12} = \mathrm{r}_{uv} \cdot n$, $h_{22} = \mathrm{r}_{vv} \cdot n$ is called the second basic quantity of surface S.

3. Weierstrass Representation of Minimal Surfaces

3.1. Weierstrass theorem

Theorem 3.1.1 [3]. The sufficient and necessary condition that the surface $S \subseteq \mathbb{R}^3$ is a minimal surface is that its parametric equation $r(u, v)$ is a harmonic function of the isothermal parameter (u, v), i. e:

$$\Delta r(u, v) = \frac{\partial^2 r(u,v)}{\partial u^2} + \frac{\partial^2 r(u,v)}{\partial v^2} = 0. \tag{13}$$

3.2. Weierstrass-Enneper Formula

Let$D \subseteq \mathbb{R}^2$ be the simply connected open region, $X: D \to \mathbb{R}^3$ be a conformally parametrized minimal surface, and(u, v)be an isothermal parameter of $X(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$then the function [11, 12]:

$$\phi_j = \frac{1}{2}\left(\frac{\partial x_j}{\partial u} - i\frac{\partial x_j}{\partial v}\right), j = 1,2,3, \tag{14}$$

is holomorphic on D and satisfies $\sum_{j=1}^{3} {\phi_j}^2 = 0, \sum_{j=1}^{3} \left|\phi_j\right|^2 \neq 0.$

If $\phi_3 \neq 0, (\phi_1, \phi_2, \phi_3) \cdot (\phi_1, \phi_2, \phi_3) = 0$, then:

$$\phi_3 = (\phi_1 - i\phi_2)\frac{\phi_1 + i\phi_2}{-\phi_3}. \tag{15}$$

Take $F = \phi_1 - i\phi_2, G = \frac{\phi_1 + i\phi_2}{-\phi_3}$, where F is a holomorphic function on D and G is a meromorphic function.

Thus, it has:

$$\begin{cases} \phi_1 = \frac{1}{2}F(1 - G^2) \\ \phi_2 = \frac{i}{2}F(1 + G^2)\cdot \\ \phi_3 = FG \end{cases} \tag{16}$$

Denote $z = u + iv$ as the complex variable in the parameter region D and introduce the complex derivative:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{17}$$

So Eq. (14) can be expressed as:

$$\frac{\partial X}{\partial z} = \left(\frac{\partial x_1}{\partial z}, \frac{\partial x_2}{\partial z}, \frac{\partial x_3}{\partial z}\right) = (\phi_1, \phi_2, \phi_3). \tag{18}$$

Eq. (16) can be expressed as:

$$\begin{cases} \frac{\partial x_1}{\partial z} = \frac{1}{2}F(1 - G^2) \\ \frac{\partial x_2}{\partial z} = \frac{i}{2}F(1 + G^2). \\ \frac{\partial x_3}{\partial z} = FG \end{cases} \tag{19}$$

Denote:

$$\Phi = \left(\frac{1}{2}F(1 - G^2), \frac{i}{2}F(1 + G^2), FG\right). \tag{20}$$

Then Eq. (19) can be written in complex vector form

$$\frac{\partial X}{\partial z} = \Phi.: \tag{21}$$

Since Φ is holomorphic on D, Eq. (21) is complete integrability, then integrating over (8) yields:

$$X = 2\mathrm{Re}\int \Phi dz = \mathrm{Re}\left(\int F(1-G^2)dz, i\int F(1+G^2)dz, 2\int FGdz\right)^T. \tag{22}$$

Eq. (22) gives the general solution of the minimal surface in $\mathbb{R}^3$, called the Weierstrass representation of the minimal surface $X(D)$. It is also commonly referred to as the Weierstrass-Enneper formula, where the functions F and G are simply the W-factors of the surface.

Note: The condition that D is simply connected is to ensure that Φ has an original function on D.

3.3. Application examples of Weierstrass representation

With the help of Eq. (14) the following example of a minimal surface can be constructed [6].

3.3.1 Catenoid

Taking $D = \mathbb{C}, F(z) = \frac{e^z}{2}, G(z) = e^{-z}$, which are two holomorphic functions defined on the complex plane $\mathbb{C}$ with zero.

It can see from the above derivation:

$$\phi_1 = \frac{1}{2}\left(\frac{e^z - e^{-z}}{2}\right), \phi_2 = \frac{i}{2}\left(\frac{e^z + e^{-z}}{2}\right), \phi_3 = \frac{1}{2}. \tag{23}$$

Since ϕ_1, ϕ_2, ϕ_3 are holomorphic functions on simply connected region D, it is known from Cauchy's theorem that they can find zero integrals on long closed paths in D, that is ϕ_1, ϕ_2, ϕ_3 have no period, and their integrals are path independent, so the calculation is given:

$$\begin{cases} x_1 = \mathrm{Re}\int F(1-G^2)dz = \mathrm{Re}\int \sinh z\, dz = \cosh u \cos v \\ x_2 = \mathrm{Re}\int iF(1+G^2)dz = \mathrm{Re}\int \mathrm{i}\cosh z\, dz = \cosh u \sin v. \\ x_3 = \mathrm{Re}\int 2FGdz = \mathrm{Re}\int 1dz = u \end{cases} \tag{24}$$

Therefore, the minimal surface is obtained as follows:

$$X(u,v) = (\cosh u \cos v, \cosh u \sin v, u)^T, \tag{25}$$

which is the catenoid. (See Figure 1).

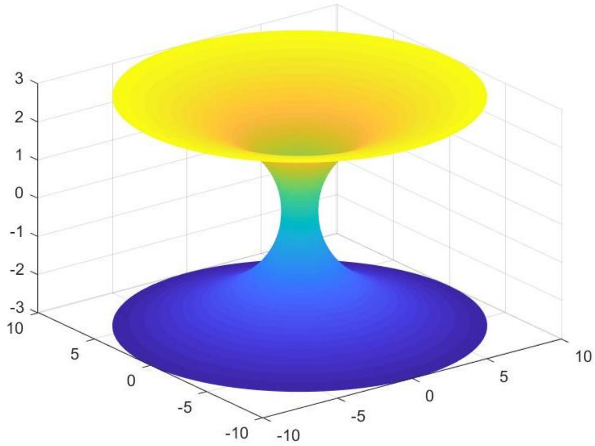

Fig. 1 Catenoid

3.3.2 Positive spiral plane

Taking $D = \mathbb{C}, F(z) = \frac{e^z}{2i}, G(z) = e^{-z}$, similarly:

$$\phi_1 = -\frac{i}{2}\left(\frac{e^z - e^{-z}}{2}\right), \phi_2 = \frac{1}{2}\left(\frac{e^z + e^{-z}}{2}\right), \phi_3 = -\frac{i}{2}. \tag{26}$$

These three functions are holomorphic on the simply connected region D, so they have no period, that is, their integration is path independent. So the following results can be obtained by direct integration.

$$\begin{cases} x_1 = \mathrm{Re}\int F(1-G^2)\mathrm{dz} = \mathrm{Re}\int -\mathrm{isinh}\, z\, dz = \sinh u \sin v \\ x_2 = \mathrm{Re}\int iF(1+G^2)\mathrm{dz} = \mathrm{Rei}\cosh z\, dz = \sinh u \cos v \\ x_3 = \mathrm{Re}\int 2FG\mathrm{dz} = \mathrm{Re}\int -i\, dz = v \end{cases}, \tag{27}$$

Therefore, the minimal surface is obtained:

$$\mathrm{X}(\mathrm{u},\mathrm{v}) = (\sinh u \sin v, \sinh u \cos v, \mathrm{v})^T. \tag{28}$$

This minimal surface is called the positive spiral surface (See Figure 2).
It is given by:

$$\mathrm{X}(\mathrm{u},\mathrm{v}) = \sinh u\,(\sin v, \cos v, 0)^T + (0,0,v)^T. \tag{29}$$

It can be seen that the positive spiral surface is a straight surface.
The functions F and G represent the first basic form and the unit normal vector of the surface.

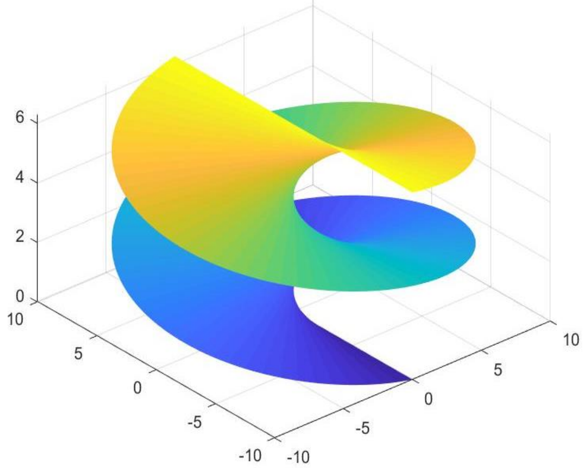

Fig. 2 Positive spiral plane

4. Representing the First and Second basic Forms of a Surface by the W-factor of a Minimal Surface

4.1. The Functions F and G are Used to Represent the First Basic Form and the Unit Normal Vector of a Surface

Theorem 4.1.1 [6,13]: Let $\mathrm{D} \subseteq \mathbb{R}^2$ be a simply connected open region, $\mathrm{X}(\mathrm{u},\mathrm{v}): \mathrm{D} \to \mathbb{R}^3$ be a minimal surface of conformal parameterization, the first basic form of $\mathrm{X}(\mathrm{u},\mathrm{v})$ is $\mathrm{I} = g_{11}(du^2 + dv^2) = g_{11}|dz|^2$, and the unit normal vector is $\mathrm{n}(\mathrm{u},\mathrm{v})$, then there are holomorphic functions F and meromorphic functions G on D such that:

(1). The first basic quantity of $\mathrm{X}(\mathrm{u},\mathrm{v})$ is expressed by:

$$g_{11} = |F|^2(1+|G|^2)^2. \tag{30}$$

and the first basic form is expressed by:

$$\mathrm{I} = |F|^2(1+|G|^2)^2|dz|^2. \tag{31}$$

(2). The unit normal vector for $\mathrm{X}(\mathrm{u},\mathrm{v})$ is:

$$\mathrm{n} = \frac{1}{1+|G|^2}(2Re\{G\}, 2Im\{G\}, |G|^2-1). \tag{32}$$

Proof: (1) From Φ in Eq. (4) and its conjugate $\overline{\Phi}$:

$$\Phi \cdot \overline{\Phi} = |\Phi|^2 = \left|\frac{\partial \mathrm{X}}{\partial \mathrm{z}}\right|^2 = \frac{1}{4}\left[\left(\frac{\partial \mathrm{X}}{\partial \mathrm{u}}\right)^2 + \left(\frac{\partial \mathrm{X}}{\partial \mathrm{v}}\right)^2\right]$$
$$= \frac{1}{4}(g_{11}+g_{22}) = \frac{1}{2}|F|^2(1+|G|^2)^2. \tag{33}$$

Then according to Eq. (7):

$$K = g_{11} = \frac{1}{2}\left[\left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial X}{\partial v}\right)^2\right] = 2|\Phi|^2 = |F|^2(1+|G|^2)^2. \tag{34}$$

Therefore,

$$I = g_{11}|dz|^2 = |F|^2(1+|G|^2)^2|dz|^2. \tag{35}$$

(2) By definition,

$$\Phi \times \overline{\Phi} = \frac{i}{2}\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}$$

$$= \frac{i}{2}|F|^2(1+|G|^2)^2(2Re\{G\}, 2Im\{G\}, |G|^2-1), \tag{36}$$

Then,

$$n = \frac{1}{g_{11}}\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = -\frac{2i}{g_{11}}\frac{\partial X}{\partial z} \times \frac{\partial X}{\partial \overline{z}} = -\frac{2i}{g_{11}}\Phi \times \overline{\Phi}$$

$$= \frac{1}{1+|G|^2}(2Re\{G\}, 2Im\{G\}, |G|^2-1). \tag{37}$$

According to the above theorem, the positive helical surface and the catenary surface have the same first basic form and unit normal vector, respectively:

$$I = |F|^2(1+|G|^2)^2|dz|^2 = \left|\frac{e^z}{2}\right|^2 (1+|e^{-z}|^2)^2|dz|^2, \tag{38}$$

and

$$n = \frac{1}{1+|e^{-z}|^2}(2Re\{e^{-z}\}, 2Im\{e^{-z}\}, |e^{-z}|^2-1). \tag{39}$$

When X is a conformal parameterized surface with mean curvature $H \equiv 0$, there is the following equivalent inscription.

Lemma 4.1.2 [6]. Let X be a conformal parameterized surface, then X is a minimal surface $\Leftrightarrow$ $X: D \to \mathbb{R}^3$ is an all-holomorphic map.

Moreover, when X is minimal, the Gauss curvature can be simply given by:

$$K = -\frac{\left|\Phi \times \frac{\partial \Phi}{\partial z}\right|^2}{|\Phi|^6}. \tag{40}$$

Proof: According to the mean curvature formula, it has:

$$2Hn = \frac{\frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2}}{g_{11}} = \frac{4\frac{\partial}{\partial z}\left(\frac{\partial X}{\partial \overline{z}}\right)}{g_{11}} = \frac{4\frac{\partial \Phi}{\partial \overline{z}}}{g_{11}}, \tag{41}$$

and thereby:

$$H \equiv 0 \Leftrightarrow \frac{\partial \Phi}{\partial \overline{z}} \equiv 0. \tag{42}$$

From Lagrange's constant equation:

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} log|\Phi|^2 = \frac{|\Phi|^2\left|\frac{\partial \Phi}{\partial z}\right|^2 - \left|\Phi \cdot \overline{\frac{\partial \Phi}{\partial z}}\right|^2}{|\Phi|^4} = \frac{\left|\Phi \times \frac{\partial \Phi}{\partial z}\right|^2}{|\Phi|^4}. \tag{43}$$

When X is a conformal parameterized surface, the Gauss curvature has the following form:

$$K = -\frac{\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} log|\Phi|^2}{|\Phi|^2}. \tag{44}$$

Substituting Eq. (43) into the above equation yields the proof.

Therefore, the following conclusions about Gauss curvature can be obtained from the above lemma.

Theorem 4.1.3 [6]. Let $D \subseteq \mathbb{R}^2$ be a simply connected open region, and $X(u,v): D \to \mathbb{R}^3$ be a minimal surface of conformal parameterization, then there exist holomorphic functions F and meromorphic functions G on D such that:

$$K = \frac{-4|G'|^2}{|F|^2(1+|G|^2)^2}. \tag{45}$$

4.2. Use Functions F and G to Represent the Second Basic Form of a Surface

Lemma 4.2.1 [13]. $X(u,v): D \to \mathbb{R}^3$ is a conformal parameterized minimal surface, where (u,v) is an isothermal parameter, and the second basic form of $X(u,v)$ is:

$$I = h_{11}du^2 + 2h_{12}dudv + h_{22}\,dv^2. \tag{46}$$

Then it has:

$$F\frac{\partial G}{\partial z} = \frac{1}{2}(h_{22} - h_{11}) + ih_{12}. \tag{47}$$

Theorem 4.2.2 [11]. Let $D \subseteq \mathbb{R}^2$ be a simply connected open region, $X(u,v): D \to \mathbb{R}^3$ be a conformal parameterized minimal surface, and (u,v) be an isothermal parameter, then:

(1) The second basic quantity of the minimal surface $X(u,v)$ is:

$$h_{11} = -h_{22} = -Re\left\{F\frac{\partial G}{\partial z}\right\}, h_{12} = h_{21} = Im\left\{F\frac{\partial G}{\partial z}\right\}; \tag{48}$$

(2) The second basic form of the minimal surface $X(u,v)$ is:

$$II = -Re\left\{F\frac{\partial G}{\partial z}dz^2\right\}. \tag{49}$$

Proof: (1) Since $X(u,v): D \to \mathbb{R}^3$ is a conformal parameterized minimal surface, then there is $g_{11} = g_{22}$, $g_{12} = 0$.

Therefore, the average curvature of the minimal surface $X(u,v)$ is:

$$H = \frac{h_{11} + h_{22}}{2g_{11}}. \tag{50}$$

From $H = 0$, it can know that $h_{11} = -h_{22}$.

From Eq. (47), it can be obtained that:

$$F\frac{\partial G}{\partial z} = -h_{11} + ih_{12}. \tag{51}$$

That is, $h_{11} = -Re\left\{F\frac{\partial G}{\partial z}\right\}, h_{12} = Im\left\{F\frac{\partial G}{\partial z}\right\}$.

(2) Derived from (1):

$$\begin{aligned} II &= h_{11}(du^2 - dv^2) + 2h_{12}dudv \\ &= Re\{(h_{11} - ih_{12})(du + idv)^2\} \\ &= -Re\left\{F\frac{\partial G}{\partial z}dz^2\right\}. \end{aligned} \tag{52}$$

Proof completed.

In summary, this section expresses the first and second basic forms of minimal surfaces in terms of holomorphic and meromorphic functions.

5. Conclusion

As a two-dimensional Riemannian manifold or a one-dimensional complex manifold, minimal surfaces have a variety of compatible mathematical structures. It has a conformal structure and Riemannian metric structure. The Weierstrass representation of minimal surface discussed in this paper is to find the general solution of the minimal surface equation using concepts and methods in complex functions such as isothermal coordinates and analytic functions. Weierstrass representation of minimal surfaces is widely used. It can not only be used to study the properties of minimal surfaces, but also to represent classical minimal surfaces. It plays a particularly prominent role in solving the Plateau problem of minimal surfaces, popularizing Bernstein's theorem, studying the stability of minimal surfaces, and finding new examples of minimal surfaces (local or global), in part because of the explicit geometric significance of the W-factor G (G is the spheroplast projection of a unit average vector). In addition, some fundamental geometric quantities of minimal surfaces can be given by W-factors. In general, only "local minimal surface" can be obtained using the differential or variational equation of minimal surface, but "local and global minimal surface" can be obtained using the Weierstrass formula of minimal surface. The Weierstrass representation of the minimal surface gives the parametric surface. It uses the isothermal parameter coordinates, and its orthogonality brings convenience for the future study of the geometric properties of the surface.

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