Applications of Algebraic Geometry in Contemporary Physics

Yang Chen

University of St Andrews, Scotland, UK

YangChenYves@163.com

Abstract. Algebraic geometry, a branch of mathematics that studies solutions to polynomial equations, has found profound applications in physics, particularly in the context of addressing singularities with significant physical implications. This article delves into the role of algebraic geometry in various areas of physics, including string theory, mirror symmetry, solitons, instanton moduli spaces, and the simplification of computational tasks such as Feynman integral reduction and multi-variable global residue computation. These applications highlight the indispensable contributions of algebraic geometry to the advancement of our understanding of the physical universe.

Keywords: Algebraic geometry, String theory, Calabi-Yau Manifold, Mirror symmetry, Scattering amplitude.

1. Introduction:

Algebraic geometry, a branch of mathematics that explores the geometric properties of solutions to polynomial equations, has emerged as a powerful tool in the realm of theoretical physics. It has led to profound discoveries and practical solutions, enhancing our comprehension of the universe and expediting complex calculations in quantum field theory. This article explores the multifaceted applications of algebraic geometry in physics, with a focus on exploring how algebraic geometry is applied in two main areas of theoretical physics: string theory and computational algebraic geometry within quantum field theory.

We delve into the theoretical underpinnings of these applications and highlight their significance in advancing our understanding of the universe. The research methodology involves an extensive review of existing literature, including research papers, textbooks, and academic articles, to gather insights into the applications of algebraic geometry in physics.

2. Singularities and Beyond:

From a broad perspective, algebraic geometry plays a crucial role in the field of physics when confronting singularities that hold substantial physical significance becomes necessary. When faced with issues involving singularities, a natural approach is to consider methods of how to achieve the resolution of these singularities. Algebraic geometry becomes particularly valuable in this context.

For instance, in the context of the Calabi-Yau Gauged Linear Sigma Model (GLSM), physical parameters, such as the Fayet-Iliopoulos (FI) parameters, exert control over the geometry of the corresponding Calabi-Yau (CY) 3-fold [1]. However, as these parameters vary to specific values, the 3-fold may develop singularities. Once the parameters deviate from these specific values, the singularities disappear, but the corresponding CY geometry and topology undergo transformations. Mirror symmetry focuses on understanding the dynamics of this process of singularity emergence and disappearance in the context of the Mirror CY.

Similarly, the open-closed string duality problem presents another scenario. When the base manifold S^3 of the Calabi-Yau T*S^3 shrinks to zero volume, a conifold singularity emerges. This singularity has two resolutions, commonly known as the deformed conifold and the resolved conifold. The former allows for the stacking of N D-branes on S^3, leading to a U(N) Chern-Simons theory, while the latter gives rise to closed topological string theory. The open/closed string duality suggests that these two distinct physical theories, stemming from different geometric treatments of the same singularity, are equivalent.

Furthermore, in various other scenarios, such as singularities in solitons, instanton moduli spaces, or Seiberg-Witten curves, singularities may arise at specific values of moduli [2]. Algebraic geometry plays a fundamental role in addressing and understanding these challenges.

3. Algebraic Geometry and String Theory:

Algebraic geometry has made significant inroads into the realm of theoretical physics, especially in string theory. By taking a closer look, it's evident that the physics applications of algebraic geometry, at least in a rudimentary sense, could trace back to the compactification of string theory on Calabi-Yau manifolds.

The solution to the Calabi conjecture, which deals with the vanishing of the Chern class, by Shing-Tung Yau provided a relatively simple algebraic geometrical characterization of such manifolds.[3] With this accomplishment, the tools and concepts inherent to algebraic geometry began to enter the purview of physicists.

Let's delve deeper into a more specific application of algebraic geometry within the context of string theory, emphasizing its tangible physical implications:

Within heterotic string theory, the existence of gauge fields as solutions to the Hermitian-Yang-Mills equations correlates with the slope stability of certain holomorphic vector bundles. This association, given the right conditions, is referred to as the Donaldson-Uhlenbeck-Yau theorem. Although the objects involved up to this point are still rooted in complex geometry, the conditions for stability can be explored within the confines of algebraic geometry. Simultaneously, the computation of cohomology for these vector bundles aligns with the number of massless chiral superfields. These calculations, furthermore, can be transformed into algebraic geometry problems through equivalence of categories, known as GAGA .

Now, turning back to Calabi-Yau manifolds, string theories formulated on two distinct Calabi-Yau manifolds can be dualized through the concept of mirror symmetry. This concept, along with related ideas like the Calabi-Yau and Landau-Ginzburg correspondences and the gauged linear sigma model, heavily employs tools from algebraic geometry.

There are other pertinent topics in the realm of string theory and quantum gravity that intertwine with algebraic geometry. For instance, consider the super-Riemann surfaces and supermoduli space. The bosonic string, which is well-defined exclusively in 26 dimensions, lends itself to an interpretation within the framework of algebraic geometry and moduli space.

Then there's F-theory, a branch of string theory, where the entire model is constructed atop elliptically fibered Calabi-Yau fourfolds. [4]This theory pays particular attention to the degenerate or singular fibers within these fourfolds.

4. Computational Algebraic Geometry in Physics:

Beyond its theoretical implications, algebraic geometry lends a helping hand to physicists grappling with the practical challenges of computation. In the realm of quantum field theory, calculating scattering amplitudes is a central task that demands intricate manipulations of complex mathematical expressions.

For example, the reduction of Feynman integrals is a formidable task within quantum field theory. These integrals, often highly complex, are central to the calculation of scattering amplitudes. K. Larsen and Y. Zhang discovered that Computational algebraic geometry offers an elegant solution by providing relatively simple Integration by Parts (IBP) reduction relations in 2015 [5]. These relations streamline the reduction process, avoiding the proliferation of powers in propagators or Gram matrix factors in the Baikov representation. In practical terms, this approach allows physicists to simplify complex integrals, avoiding unnecessary computational complexity. The use of computational algebraic geometry in this context accelerates the calculation of scattering amplitudes, contributing to more efficient and precise results.

Another computational challenge arises when calculating global residues of complex functions, a problem encountered in various physical contexts, including the Cachazo-He-Yuan (CHY) representation. Traditional methods involve solving for the position of each root and then summing up the residues. However, for complex functions, this approach becomes computationally cumbersome. Here, techniques from algebraic geometry like the Bezoutian matrix enables physicists to calculate global residues directly without the need to solve for the roots of multivariate polynomials. In 2015, M. Sogaard and Y. Zhang employed this approach to simplify the calculation of amplitudes within the CHY formalism [6]. This methodology ensures that the results obtained are guaranteed to be rational functions, thus streamlining the computation process.

This computational advantage is particularly valuable in the context of the CHY formula, which itself poses a global residue problem. By leveraging algebraic geometry, physicists can navigate the intricate web of complex functions more efficiently, ensuring that their calculations are both accurate and manageable.

5. Conclusion

Algebraic geometry's journey into the realm of theoretical physics has transformed our understanding of the universe's fundamental forces. From its foundational role in string theory and mirror symmetry to its indispensable contributions in addressing singularities and simplifying computational tasks, algebraic geometry stands as a testament to the unbreakable bond between mathematics and the physical world. It offers profound insights and elegant solutions to several important questions in theoretical physics. In addition to its direct applications, it's worth noting that string theory, in itself, provides a wealth of valuable mathematical problems closely related to algebraic geometry. The mathematical intricacies arising within string theory inspire mathematicians to explore new frontiers in algebraic geometry, such as the recent established connection between the geometric Langlands correspondence and S-duality.[7] This reciprocal relationship between string theory and algebraic geometry fuels ongoing research, pushing the boundaries of both fields.

Future research in this dynamic field should focus on further exploring the mathematical connections between algebraic geometry and physics. Additionally, developments in computational algebraic geometry techniques can lead to even more efficient tools for physicists. Exploring applications of algebraic geometry in emerging areas of physics, such as quantum computing and quantum information theory, holds promise for expanding our understanding of the universe. Collaboration between mathematicians and physicists will continue to be essential in unlocking the mysteries of the cosmos through algebraic geometry.

References

[1] (2009) C. Closset, Toric geometry and local Calabi-Yau varieties: An introduction to toric geometry (for physicists)

[2] (2003) H. Nakajima, K. Yoshioka, Lectures on Instanton Counting, 7-17

[3] (1978) Yau, Shing-Tung, "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I", Communications on Pure and Applied Mathematics, 31 (3): 339–411

[4] (2015) W. Taylor, Y. Wang, The F-theory geometry with most flux vacua, Journal of High Energy Physics

[5] (2015) K. Larsen, Y. Zhang, Integration-by-parts reductions from unitarity cuts and algebraic geometry, Physical Review D 93(4).

[6] (2015) M. Sogaard, Y. Zhang, Scattering Equations and Global Duality of Residues, Physical Review D 93(10).

[7] (2006) A. Kapustin, E. Witten, Electric-Magnetic Duality and The Geometric Langlands Program, Communications in Number Theory and Physics 1(1).