

Analysis of Chaos of Chen Equation

Zhihui Zhao*

Department of Mathematics, King's College London, London, United Kingdom

*Corresponding author: dongzhaoyu@edu.iwhr.com

Abstract. As a matter of fact, chaos was first discovered by Poincaré. Then, the Edward Lorenz observed by chance the chaotic phenomenon when he predicted the weather. More powerful computers greatly aided chaos theory in recent years. With this in mind, this study analyses the chaos of Chen equations, the stability of Chen system, and the sensitive of parameters based on the staggered methods. The instability of the system is obtained by Routh-Hurwitz criterion. In addition, the numerical solution is obtained by Runge-Kutta method. Then, one draws the Chen attractors. Finally, the sensitive analysis in a new way implies that the solution is sensitive to the initial values, and the sensitivity relationship between the same component to different parameters is linear, but that between the different components to the same parameter is chaotic, and the different component to different parameters is also chaotic. This study is a multi-dimensional and comprehensive analysis of sensitivity analysis.

Keywords: Chaos system; Chen system; stability analysis; sensitive system.

1. Introduction

The first chaotic phenomenon was found when Poincaré studied the restricted three-body problem, in 19th century. He found that the trajectory of the particle is non-periodic and neither monotonous nor tends to a fixed point [1-3]. Afterwards, Jacques Hadamard found that the relationship between this instability and Lyapunov exponent, that is, all orbits are unstable and their maximum Lyapunov exponents are positive. Because the chaotic system involves the iteration of mathematical formula, it depends on the development of computer to a great extent. Hence more powerful computers greatly aided chaos theory. In 1961, Edward Lorenz discovered another property of chaotic system, initial value sensitivity. He observed by chance the chaotic phenomenon in his work of predicting the weather [4]. When he and his partners Ellen Fetter and Margaret Hamilton used a simple digital computer to calculate weather prediction models, they got an entirely different result due to the lack of the computer accuracy, which means that for long-terms results, a tiny perturbation in the initial value condition can completely change the results. The term “chaos” was coined by American mathematicians York and Li [5]. In 1976, American ecologist Robert studied the one-dimensional square map, and pointed out that the one-dimensional iterative map Logistic map can also produce complex period doubling and chaotic motion [6]. American physicist Feigenbaum established the concept of renormalization group and uncovered the scaling and universal constants in the period-doubling bifurcation phenomenon in 1978, giving chaos a strong theoretical basis in contemporary science [7]. The notion of chaos has been extensively and in-depth researched since the 1980s.

Academics have concentrated on the features and attributes of chaos as well as how the system transitions from order to fresh disorder. In 1983, the Chua's circuit invented by Chua showed rich chaotic behavior due to its simple structure and easy implementation. Chua's circuit is one of the earliest chaotic models. It has rich dynamic behaviors, such as period-doubling bifurcation, transformation of scroll structure, and typical chaos. The high-dimensional hyperchaotic constructed in Chua's circuit is also a research hotspot. In recent years, scholars have focused on chaos theory research on numerical calculation methods, using the Lyapunov exponents to determine whether a system has chaotic phenomenon, sensitivity to initial conditions, synchronization, etc. Previous study also analyses the dynamic behavior of Chen oscillator mathematically and numerically [8]. The system's bifurcation plot, eigenvalues, equilibrium graph, and Lyapunov exponents were all examined by the author. The findings demonstrate that the Chen system is chaotic. In addition, the importance of factors such initial conditions and integral step size in the evaluation of the Lyapunov exponents

is also discussed. As a matter of fact, the accuracy and reliability of the Lyapunov exponents depends on these factors. In other words, the initial values determine the starting point of the system trajectory, so different initial conditions can lead to different trajectories, which lead to different Lyapunov exponents. A small integration step can lead to more accurate results, but it also increases computational costs. Global stability of Liu system is analysed in paper [9]. Sensitivity analysis is also an indispensable research topic. Akinlar gives an original approach [10]. The derivative of each component for each parameter is obtained by the staged methods, and the relationship between them is observed. In this paper, firstly the stability analysis of Chen system is carried out, and one gets the conclusion that Chen system is unstable. Next, using the Runge-Kutta method, the numerical solution of the system is computed, and MATLAB is used to obtain the double-scroll attractors. Finally, the sensitivity analysis is carried out by Akinlar method.

2. Methodology

The Chen system is as follows:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy \\ \dot{z} = xy - bz \end{cases} \quad (1)$$

where a, b and c are real parameters. This study uses the Lyapunov exponent to check that the system is chaotic when $a = 40$, $b = 3$ and $c = 25$.

2.1. Stability Analysis

Judging whether the system is stable depends on whether the solution of equations whose derivative is zero is stable, then one rewrites the system as follows

$$\begin{cases} f(x, y, z) = a(y - x) = 0 \\ g(x, y, z) = (c - a)x - xz + cy = 0 \\ h(x, y, z) = xy - bz = 0 \end{cases} \quad (2)$$

Since the system is autonomous (i.e., the right hand side of the equations do not contain the time variable t), the root of the algebraic equations (2) are $(\tilde{x}, \tilde{y}, \tilde{z})$ called the equilibrium point (singular solution) of the equations. If all possible initial conditions (x_0, y_0, z_0) satisfy

$$\begin{cases} \lim_{t \rightarrow 0} x(t) = x_0 \\ \lim_{t \rightarrow 0} y(t) = y_0 \\ \lim_{t \rightarrow 0} z(t) = z_0 \end{cases} \quad (3)$$

the singular solution is said to be stable, otherwise the root is said to be unstable. Then one takes the Eq. (1) and Eq. (2) to get three solutions $X_1 = (0, 0, 0)$ or $X_{2,3} = (\pm\sqrt{b(2c-a)}, \pm\sqrt{b(2c-a)}, 2c-a)$. Next, one finds the characteristic roots and characteristic equations of the equation. First, this study will carry out Taylor expansion of f , g and h at the equilibrium point. Since $x - \bar{x}$ is infinitesimal at second order and above, only the linear part is retained here. Next, one replaces $x - \bar{x}$ with a new variable $\bar{x}$, that is, one translates the function $\bar{x}$ units, and the properties have not changed. Therefore, this paper mainly studies this equation after translation. One writes the equations as matrix form

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}' = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = A \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \quad (4)$$

where A is the Jacobian matrix. Then for the solution X_1 , the eigenfunction is

$$f_1(\lambda) = |A - \lambda I| = -\lambda^3 + (c - a - b)\lambda^2 + (2ab + 2ac + bc - a^2)\lambda - a^2b \quad (5)$$

Likewise, one gets the eigenfunction of the solution X_2

$$f_2(\lambda) = |A - \lambda I| = -\lambda^3 + (c - a - b)\lambda^2 - bc\lambda + 2a^2b - 4abc \quad (6)$$

and the eigenfunction of the solution X_3 is

$$f_3(\lambda) = |A - \lambda I| = -\lambda^3 + (c - a - b)\lambda^2 - bc\lambda + 2a^2b - 4abc \quad (7)$$

Case 1. If the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A are all real, the solution of equation (5) is

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = C_1 T_1 e^{\lambda_1 t} + C_2 T_2 e^{\lambda_2 t} + C_3 T_3 e^{\lambda_3 t} \quad (8)$$

It is obvious that $(\bar{x}, \bar{y}, \bar{z})^T \rightarrow 0$, when $\lambda_i < 0$ for all $i = 1, 2, 3$.

Case 2. If the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ have complex numbers, one assumes that $\lambda_1 = \alpha_1 + i\beta_1, \lambda_2 = \alpha_2 + i\beta_2$. The solution is

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = C_1 T_1 e^{\alpha_1 t} \cos \beta_1 t + C_2 T_2 e^{\alpha_2 t} \cos \beta_2 t + C_3 T_3 e^{\lambda_3 t} \quad (9)$$

Since $|\cos \beta_i t| \leq 1$, $(\bar{x}, \bar{y}, \bar{z})^T \rightarrow 0$, when $\lambda_3 < 0$ and $\alpha_i < 0$ for all $i = 1, 2$. In summary, when the eigenvalues have a negative real part, the system is stable.

2.2. Hurwitz Criterion

Routh-Hurwitz criterion is a criterion for polynomial equations with positive real roots. It determines whether the polynomial equation has unstable roots by algebraic operation of polynomial coefficients.

Definition (Hurwitz determinant). It is composed of the coefficients of the polynomial equation according to the following rules: the main diagonal elements are composed of a_{n-1} to a_0 in turn, and each row is based on the coefficients on the main diagonal, and the subscripts of the coefficients are decrease in turn to the left; the subscripts of the coefficients increase in turn to the right. If the subscript is greater than n or less than 0 , the coefficient is zero. That is to say, the elements of the k th row take the adjacent items in the sequence $(0, \dots, 0, a_0, \dots, a_k, \dots, a_n, 0, \dots, 0)$ with the element a_k on the diagonal as the center. For a polynomial equation $a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$, all the Hurwitz determinant is

$$\Delta_1 = a_2, \quad \Delta = \begin{vmatrix} a_2 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \Delta = \begin{vmatrix} a_2 & a_3 & 0 \\ a_0 & a_1 & a_2 \\ 0 & 0 & a_0 \end{vmatrix} \quad (10)$$

Theorem (Hurwitz criterion). The real parts of all roots of a polynomial equation less than 0 if and only if all coefficients of the polynomial equation are greater than 0, that is $a_i > 0$ for all $i = 1, 2, \dots, n$, and also all the Hurwitz determinant is bigger than 0, which means $\Delta_i > 0$ for all $i = 1, 2, \dots, n$. Obviously, the coefficient $a_3 = -1 \neq 0$, therefore, the system is not stable.

Remark. If any on of these conditions $a_1 = 0, a_3 = 0, a_1 a_2 - a_3 = 0$ is satisfied, the Chen system is in a critical state [11, 12].

2.3. Solving and Parameter Range

For Chen systems, the numerical solution is found using the Runge-Kutta method. For a Cauchy problem

$$\begin{cases} \frac{dx}{dt} = f(x, t) \\ x(t_0) = x_0 \end{cases} \quad (11)$$

and a step $h > 0$, which means $t_{n+1} = t_n + h$, the approximation of $x(t_{n+1})$ is x_{n+1} , given by x_n and h . Then the form is as follows

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (12)$$

for $n = 0, 1, 2, \dots$, where k_n is the tangent slop of f at each quarter point of x_i, x_{i+1} . Then the Runge-Kutta method of Chen systems is shown as following.
For x , one has:

$$\begin{cases} x_{n+1} = x_n + \frac{h}{6}(k_{11} + 2k_{12} + 2k_{13} + k_{14}) \\ k_{11} = a(y_n - x_n) \\ k_{12} = a[y_n + \frac{h}{2}k_{11} - (x_n + \frac{h}{2}k_{11})] \\ k_{13} = a[y_n + \frac{h}{2}k_{12} - (x_n + \frac{h}{2}k_{12})] \\ k_{14} = a[y_n + hk_{13} - (x_n + hk_{13})] \end{cases} \quad (13)$$

For y , one has:

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(k_{21} + 2k_{22} + 2k_{23} + k_{24}) \\ k_{21} = (c - a)x_n - x_n z_n + cy_n \\ k_{22} = (c - a)(x_n + \frac{h}{2}k_{21}) - (x_n + \frac{h}{2}k_{21})(z_n + \frac{h}{2}k_{21}) + c(y_n + \frac{h}{2}k_{21}) \\ k_{23} = (c - a)(x_n + \frac{h}{2}k_{22}) - (x_n + \frac{h}{2}k_{22})(z_n + \frac{h}{2}k_{22}) + c(y_n + \frac{h}{2}k_{22}) \\ k_{24} = (c - a)(x_n + hk_{23}) - (x_n + hk_{23})(z_n + hk_{23}) + c(y_n + hk_{23}) \end{cases} \quad (14)$$

Then for z one has:

$$\begin{cases} z_{n+1} = z_n + \frac{h}{6}(k_{31} + 2k_{32} + 2k_{33} + k_{34}) \\ k_{31} = x_n y_n - b z_n \\ k_{32} = (x_n + \frac{h}{2} k_{31})(y_n + \frac{h}{2} k_{31}) - b(z_n + \frac{h}{2} k_{31}) \\ k_{33} = (x_n + \frac{h}{2} k_{32})(y_n + \frac{h}{2} k_{32}) - b(z_n + \frac{h}{2} k_{32}) \\ k_{34} = (x_n + h k_{33})(y_n + h k_{33}) - b(z_n + h k_{33}) \end{cases} \quad (15)$$

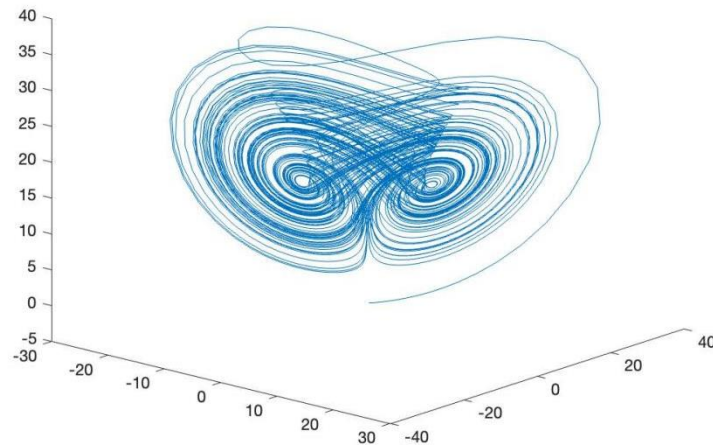


Fig. 1 Chen attractor with parameters $a = 40$, $b = 3$ and $c = 28$

3. Results and Discussion

For the initial values $x_0 = -0.1$, $y_0 = 0.5$ and $z_0 = -0.6$, by using Runge-Kutta method Chen attractors are shown as Figure 1. Since one just focuses on the parameters, one fixes the initial conditions $x_1(0) = -0.1$, $x_2(0) = 0.5$ and $x_3(0) = -0.6$. Then according to Akinlar [10], a new variable m is defined as $m = (x_1, x_2, x_3, s_1, s_2, s_3)^T$, where

$$\begin{cases} s_1 = \frac{dx}{da} = \left(\frac{dx_1}{da} \quad \frac{dx_2}{da} \quad \frac{dx_3}{da} \right)^T = (s_{11} \quad s_{12} \quad s_{13})^T \\ s_2 = \frac{dx}{db} = \left(\frac{dx_1}{db} \quad \frac{dx_2}{db} \quad \frac{dx_3}{db} \right)^T = (s_{21} \quad s_{22} \quad s_{23})^T \\ s_3 = \frac{dx}{dc} = \left(\frac{dx_1}{dc} \quad \frac{dx_2}{dc} \quad \frac{dx_3}{dc} \right)^T = (s_{31} \quad s_{32} \quad s_{33})^T \end{cases} \quad (16)$$

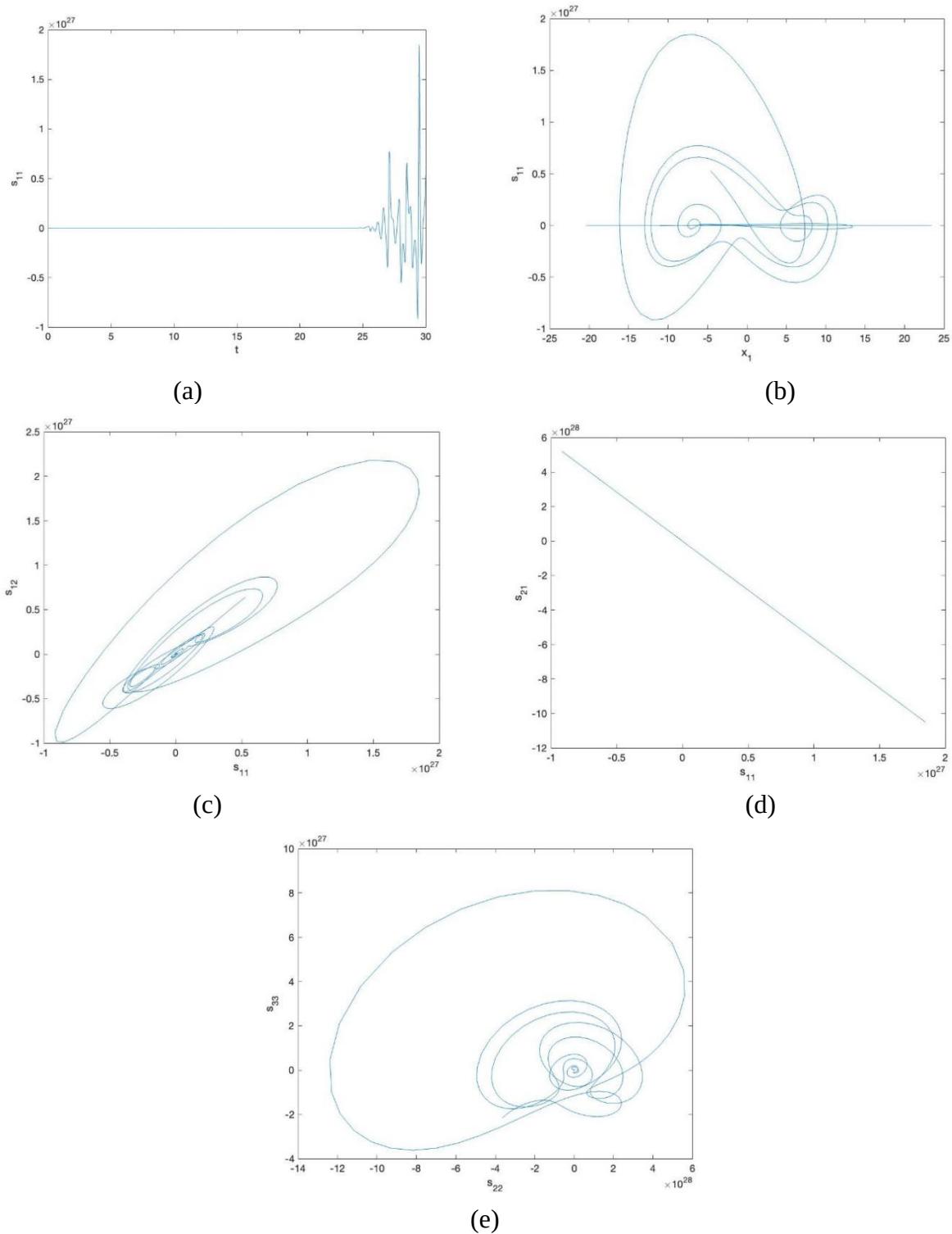


Fig. 2 Some relationships between components

Finally, the sensitive equations are as follows:

$$\begin{pmatrix} \dot{s}_{11} \\ \dot{s}_{12} \\ \dot{s}_{13} \\ \dot{s}_{21} \\ \dot{s}_{22} \\ \dot{s}_{23} \\ \dot{s}_{31} \\ \dot{s}_{32} \\ \dot{s}_{33} \end{pmatrix} = \begin{pmatrix} x_2 - x_1 + a(s_{12} - s_{11}) \\ (c - a - x_3)s_{11} + cs_{12} - x_1s_{13} - x_1 \\ x_2s_{11} + x_1s_{12} - bs_{13} \\ a(s_{22} - s_{21}) \\ (c - a - x_3)s_{21} + cs_{22} - x_1s_{23} \\ x_2s_{21} + x_1s_{22} - bs_{23} - x_3 \\ a(s_{32} - s_{31}) \\ (c - a - x_3)s_{31} + cs_{32} - x_1s_{33} + x_1 + x_2 \\ x_2s_{31} + x_1s_{32} - bs_{33} \end{pmatrix} \quad (17)$$

with the initial conditions $s_{ij} = 0$ for all $i, j = 1, 2, 3$.

The numerical solution of the sensitive equation is obtained by Runge-Kutta method with the parameters $a = 40$, $b = 3$ and $c = 25$. One has obtained a lot of diagrams, and only a few representative ones are shown here. Figure 2(a) is the phase portrait of s_{11} , and remember that it represents the component the sensitivity of x_1 , which means that the x_1 is highly depend on the initial conditions. Moreover, Figure 2(b) confirms this fact, that. The phase diagrams of the other eight components are similar. Hence for each parameter, a tinny perturbation can lead to entirely different results. Furthermore, the relationship between the sensitivity of each component to each parameter is also worthy of attention. Seen from Figure 2(c), the sensitivity of x_2 and x_1 to parameter a is nonlinear, and the sensitivity of all different components to the same parameter is the same. In addition, Figure 2(d) describes an interesting property, that is, s_{21} to s_{11} is linear, that is to say, as long as the change of a and b satisfies a certain relationship, the component x_1 can be unchanged. Finally, Figure 2(e) illustrates that changing both c and b can completely change the results.

4. Limitation & Future Outlooks

There are many ways to find numerical solutions to differential equations, such as Euler's method, Backward Euler method, First-order exponential integrator method and Runge-Kutta method, etc, but they all have their own limitations. For instance, the value of x_{i+1} in the Runge-Kutta method only uses x_i value, but this is obviously not precise enough, and the value of x_{i-1} or even x_{i-2} can be used to expand the volume of the input value, which will make the value of x_{i+1} more accurate. And the accuracy of the Runge-Kutta method also depends on the step h , if h is small enough, the solution will tend to be the exact solution, but this will increase the volume of the calculation, moreover, the increase of the number of recursive steps in the computer will accumulate errors, making the final result more inaccurate. So how to find the balance between these two factors is material. Likewise, the accuracy of Lyapunov exponent also depends on the length of the iteration time. Finally, as for the sensitivity analysis method of Akinlar, since the method discusses the sensitivity of the parameters at a fixed parameter point, it is not convenient to see the sensitivity of the solution of the differential equation to the parameters in a comprehensive way. This study, of course, only addresses a particular system; thus, the approach might not apply to other systems. In the future, to find more accurate solutions of the differential equations one can use the Adams-Bashforth methods, which is more precise and the computational cost is smaller than Runge-Kutta.

5. Conclusion

In summary, a relatively comprehensive analysis of the Chen system is carried out, including analyzing the stability using the eigenvalues and eigenfunctions and Routh-Hurwitz criterion, solving its numerical solution by Runge-Kutta method, obtaining the double-scroll attractor, and analyzing the sensitivity of these parameters and the relationship between these sensitivities. Nevertheless, it also has some limitations, the biggest one is lack of the accuracy. It is hoped to use a more precise method to analyze later.

References

- [1] Cheng Q, Sun C, Zhang X, et al. Short-Term load forecasting model and method for power system based on complementation of neural network and fuzzy logic. *Transactions of China Electrotechnical Society*, 2004, 19(10): 53-58.
- [2] Poincaré J H. The three-body problem and the equations of dynamics: Poincaré's foundational work on dynamical systems theory. Popp, Bruce D. (Translator). Cham, Switzerland: Springer International Publishing, 2017.
- [3] Diacu F, Holmes P. *Celestial Encounters: The Origins of Chaos and Stability*. Princeton University Press, 1996.
- [4] Lorenz E N. Deterministic non-periodic flow". *Journal of the Atmospheric Sciences*, 1963, 20(2): 130–141.
- [5] Li T, Yorke J A. Period Three Implies Chaos, *American Mathematical Monthly*, 1975, 82: 985.
- [6] May R. Simple mathematical models with very complicated dynamics. *Nature*, 1976, 261: 459–467.
- [7] Feigenbaum M J. Quantitative universality for a class of nonlinear transformations. *Journal of Statistical Physics*, 1978, 19(1): 25–52.
- [8] Nuñez-Perez J C, Adeyemi V A, Sandoval-Ibarra Y, et al. Mathematical and numerical analysis of the dynamical behavior of Chen oscillator. *International Journal of Dynamics and Control*, 2020, 8: 386-395.
- [9] Daftardar-Gejji V, Bhalekar S. Chaos in fractional ordered Liu system. *Comput Math Appl*, 2010, 59(3): 1117-1127.
- [10] Mehmet A A. A New Method for Parameter Sensitivity Analysis of Lorenz Equations. *Mathematical Problems in Engineering*, 2013: 537207.
- [11] Nuñez-Perez J C, Adeyemi V A, Sandoval-Ibarra Y, et al. Mathematical and numerical analysis of the dynamical behavior of chen oscillator. *Int. J. Dynam. Control*, 2020, 8: 386–395.
- [12] Elsgolts L E, Yankovsky G. *Differential equations and the calculus of variations*. Mir, Moscow, 1977.