A Contour Integral and Complex Power Series Proof of Stirling Formula

Zhiqi Zhang*

Department of Art and Science, University of Toronto, Toronto, Canada

*Corresponding author: zhiqiha.zhang@mail.utoronto.ca

Abstract. As a matter of fact, the Stirling formula is a very important formula for estimating the size of factorials, which effectively simplifies the calculation of factorials. Based on the convergency theorem, it is very accurate when n is very small, for example, when n=6, the error is only 1.4%. This formula was first discovered by Abraham de Moivre and Stirling, and mathematicians such as provided much proof of it. In addition, there is a famous proof that only relies on ordinary calculus. With this in mind, this paper attempts to independently solve this problem using simple complex analysis methods. To be specific, contour integral, Residue Theorem, complex series will be demonstrated directly and immediately. At the same time, the proof processing will be presented in detail based on the derivations of the formulae. In the meantime, the current limitations will be clarified and the prospects will be proposed according to the analysis. Overall, these results shed light on guiding further exploration of Stirling Formula proofing and applications.

Keywords: Stirling formula; Euler; contour integral; residue theorem; complex series.

1. Introduction

The Stirling formula is an important estimation formula that can accurately estimate the values of factorials or gamma functions at far points. There is much proof about it, Euler has provided proof of this formula, which uses differential equations extended to infinite orders [1]. Other proofs rely on the Wallace product formula [2]. For the Euler McLaughlin summation formula. In fact, this problem can easily be transformed into an infinite series problem. The problem of summing infinite series can often be solved by constructing complex functions and using the Residue Theorem. For example, the Basel problem, which one is very familiar with, is to find the sum of the reciprocal squares of all natural numbers (i.e., ζ (2)). Euler provided proof in 1734 [3]. His skill in solving this problem is breathtaking, visually appealing, but difficult to reproduce. This problem can be easily solved through the residue theorem with minimal computational complexity. In fact, the values of Riemann ζ function at positive and even numbers can be obtained from the residue theorem. This example demonstrates that using the residue theorem to calculate a series can greatly weaken the requirement for skill and avoid significant computational complexity [4]. The proof method presented in this paper is only based on residue theorem and uses complex series to make certain estimates, without resorting to some other formulas or second-order conclusions. The idea behind this proof is very direct, which is conducive to understanding the Stirling formula from a more fundamental level.

2. Proof of Theorem

The Stirling Theorem can be described as:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, n \to \infty \tag{1}$$

The statement above means:

$$\lim_{n\to\infty} \frac{n^{n+\frac{1}{2}}}{n!\, e^n} = \frac{1}{\sqrt{2\pi}} \tag{2}$$

Taking log for both sides, one has:

$$\lim_{n\to\infty}[(n+\frac{1}{2})logn - \sum_{k=1}^{n} logK - n] = -\frac{log(2\pi)}{2} \tag{3}$$

One can rewrite the L.H.S. to the sum form, let

$$S_n = \left(n+\frac{1}{2}\right)logn - \sum_{k=1}^{n} log\,k - n \tag{4}$$

which is sum of some $\{a_n\}$ and $a_1 = S_1 = -1$

$$a_{n+1} = S_{n+1} - S_n = \left(n+\frac{1}{2}\right)log\left(1+\frac{1}{n}\right) - 1, n \geq 1 \tag{5}$$

Therefore, one only needs to prove that

$$\sum_{n=1}^{\infty}\left[\left(n+\frac{1}{2}\right)log\left(1+\frac{1}{n}\right)-1\right] = 1 - \frac{log(2\pi)}{2} \tag{6}$$

Considering:

$$f(z) = \left[\left(z+\frac{1}{2}\right)log\left(1+\frac{1}{z}\right) - 1\right]cot(\pi z) \tag{7}$$

and take log on the principal branch, consider the contour (shown in Fig. 1), one can see that, $\forall k \in \mathbb{N},\ k \neq 0, -1$, then:

$$\begin{aligned} Res(f,k) &= \lim_{z\to k}(z-k)f(z) = \lim_{z\to k}\left[\left(z+\frac{1}{2}\right)log\left(1+\frac{1}{z}\right)-1\right]\lim_{z\to k}(z-k)\,cot(\pi z) \\ &= \frac{1}{\pi}\left[\left(k+\frac{1}{2}\right)log\left(1+\frac{1}{k}\right)-1\right] \end{aligned} \tag{8}$$

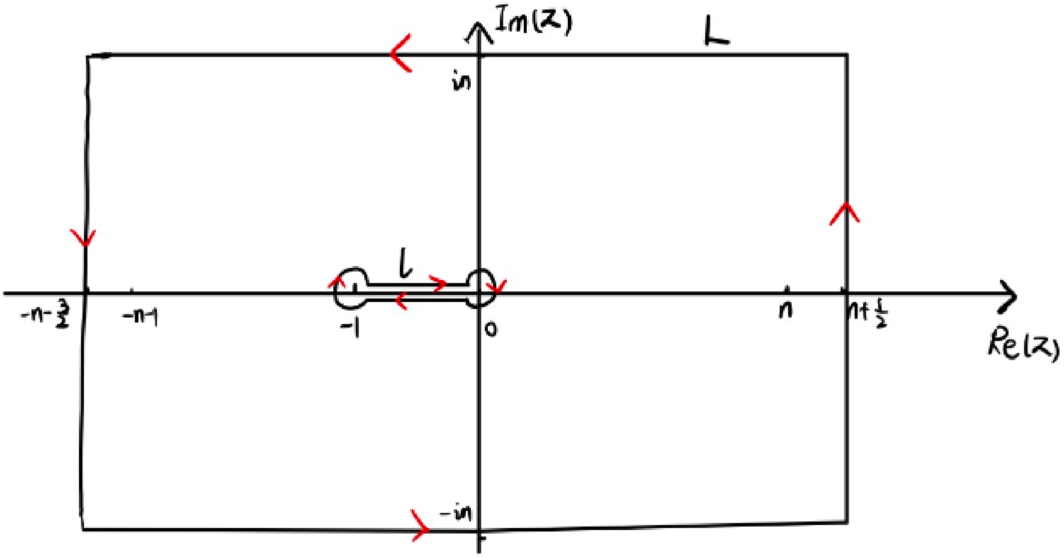

Fig. 1 Contour used for integrating f, including L and ℓ.

For $k \geq 1$, one discovers that:

$$Res(f,-k-1) = \frac{1}{\pi}\left[\left(-k-\frac{1}{2}\right)log\left(\frac{k}{k+1}\right)-1\right] = \frac{1}{\pi}\left[\left(k+\frac{1}{2}\right)log\left(1+\frac{1}{k}\right)-1\right] \tag{9}$$

so,

$$Res(f,-k-1)=Res(f,k) \tag{10}$$

Using Residue Theorem on Figure 1:

$$\int_L^{\prime} f(z)\,dz+\int_\ell^{\prime} f(z)\,dz \;=\; 2\pi i\left[\sum_{k=-n-1}^{-2} Res(f,k)+\sum_{k=1}^{\mathrm{n}} Res(f,k)\right]=4\pi i\sum_{k=1}^{n} Res(f,k) \tag{11}$$

First, one considers $\int_L^1 f(z)\,dz$. when |z| is large enough,

$$\left(z+\frac{1}{2}\right)\log\left(1+\frac{1}{z}\right)-1=\left(z+\frac{1}{2}\right)\left(\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}+\frac{1}{3}\cdot\frac{1}{z^3}-\cdots\right)-1=\frac{1}{12}\cdot\frac{1}{z^2}+o\left(\frac{1}{z^2}\right) \tag{12}$$

so when $z\ \rightarrow \infty$:

$$\left(z+\frac{1}{2}\right)\log(1+\frac{1}{z})-1\sim\frac{1}{12}\cdot\frac{1}{z^2} \tag{13}$$

One already knows that $when\, n\ \rightarrow \infty$

$$\int_L^{\prime}\left|\frac{\cot(\pi z)}{z^2}\right||dz|\rightarrow 0 \tag{14}$$

which is easy to check. Then one can conclude that $when\, n\ \rightarrow \infty$:

$$\left|\int_L^{\prime} f(z)\ dz\right| \leq \int_L^{\prime}|f(z)||dz|\rightarrow 0 \tag{15}$$

Now, one takes $n\ \rightarrow\ \infty$, one has:

$$\sum_{k=1}^{\infty} Res(f,k)=\frac{1}{4\pi i}\int_\ell^{\prime} f(z)\,dz \tag{16}$$

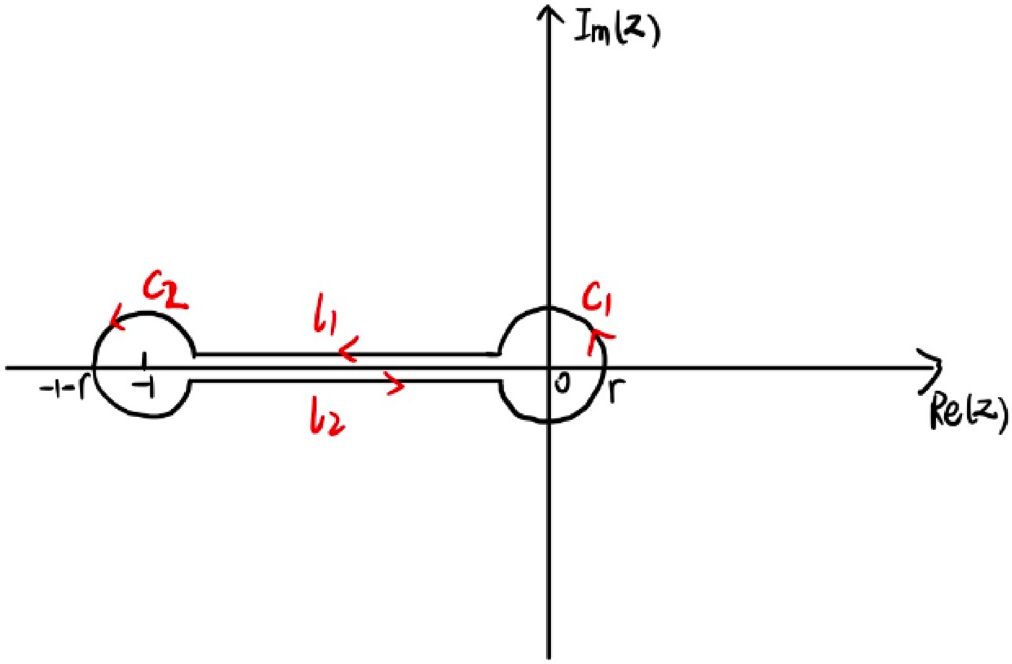

Fig. 2 The four parts of contour ℓ (in the negative direction)

Then consider $\int_\ell^{\prime} f(z)\,dz$ (seen from Fig. 2), for C_1, one writes

$$\log\left(1+\frac{1}{z}\right)=\log(1+z)-\log z \tag{17}$$

and takes log on principal branch:

$$\cot(\pi z)=\frac{1}{\pi z}-\frac{\pi z}{3}-\cdots \tag{18}$$

One has

$$\lim_{z\to 0}\left(\cot(\pi z)-\frac{1}{\pi z}\right)=0 \tag{19}$$

One also knows that

$$\lim_{z\to 0} z\ \log z\ =\ 0 \tag{20}$$

Then, one derives

$$\lim_{r\to 0}\int_{C_1}^{\prime}\left[\left(z+\frac{1}{2}\right)(\log(1+z)-\log z)-1\right]\cdot\left[\cot(\pi z)-\frac{1}{\pi z}\right]dz$$
$$=2\pi i\ \lim_{z\to 0} z\cdot\left[\left(z+\frac{1}{2}\right)\ (\log(1+z)-\ \log z)-1\right]\cdot\ \left[\cot(\pi z)-\frac{1}{\pi z}\right]=0 \tag{21}$$

Thus, one can just consider

$$\int_{C_1}^{\prime}\frac{1}{\pi z}\left[\left(z+\frac{1}{2}\right)(\log(1+z)-\log z)-1\right]dz \tag{22}$$

which means one replaces cot $(\pi \mathrm{z})$ to $\frac{1}{\pi z}$. This simpler integration is equal to:

$$\int_{C_1}^{\prime}\frac{\left(z+\frac{1}{2}\right)\log(1+z)}{\pi z}\,dz-\int_{C_1}^{\prime}\frac{\log z}{\pi}\,dz-\int_{C_1}^{\prime}\frac{\log z}{2\pi z}\,dz-\int_{C_1}^{\prime}\frac{1}{\pi z}\,dz \tag{23}$$

For which

$$\int_{C_1}^{\prime}\frac{\left(z+\frac{1}{2}\right)\log(1+z)}{\pi z}\,dz=\ 0 \tag{24}$$

$$\lim_{r\to 0}\int_{C_1}^{\prime}\frac{\log(z)}{\pi}\,dz=2\pi i\ \lim_{z\to 0} z\,logz\ =\ 0 \tag{25}$$

$$\int_{C_1}^{\prime}\frac{logz}{2\pi z}dz=\frac{1}{2\pi}\left[\frac{1}{2}\ log^2(z)\right]_{C_1}=\frac{1}{4\pi}\ [(\log r+\ \pi i)^2-\ (\log r-\ \pi i)^2]=\ i\log r \tag{26}$$

$$\int_{C_1}^{\prime}\frac{1}{\pi z}\,dz=\frac{1}{\pi}\cdot 2\pi i=2i \tag{27}$$

One has:

$$\int_{C_1}^{\prime}f(z)dz=\ -i\ (\log r+2)+o(1) \tag{28}$$

Here, $o(1)$ represents a term about r which approaches to zero when $r\to 0$. Then, one considers C_2, let $z=\ -1-\zeta,\ then\ dz=\ -d\zeta$. When path of ζ is C_1, the path of z is C_2.

$$\int_{C_2}' f(z)dz = -\int_{C_1}' f(-1-\zeta)d\zeta = -\int_{C_1}'\left[\left(-\zeta-\frac{1}{2}\right)log\left(\frac{\zeta}{1+\zeta}\right)-1\right]cot(-\pi-\pi\zeta)\,d\zeta$$

$$= \int_{C_1}'\left[\left(\zeta+\frac{1}{2}\right)log\left(\frac{1+\zeta}{\zeta}\right)-1\right]cot(\pi\zeta)\,d\zeta = \int_{C_1}' f(\zeta)d\zeta = -i\,(\log r+2)+o(1) \qquad (29)$$

Considering ℓ_1, ℓ_2, one writes $\log(1+\frac{1}{z}) = \log(1+z) - \log(z)$, take log on principle branch. The value of arg z on ℓ_1 is π, on ℓ_2 is $-\pi$, so the difference between the values of $\log(1+\frac{1}{z})$ on ℓ_2 and ℓ_1 is $2\pi i$. Thus, one has:

$$\int_{\ell_1}' f(z)dz + \int_{\ell_2}' f(z)dz = \int_{-1+r}^{-r} 2\pi i\left(x+\frac{1}{2}\right)\cot(\pi x)\,dx$$

$$= 2i\left[(1-2r)\log(\sin\pi r) - \frac{1}{\pi}\int_{\pi r}^{\pi(1-r)}\log(\sin x)\,dx\right] \qquad (30)$$

$$\lim_{r\to 0}\int_{\pi r}^{\pi(1-r)}\log(\sin x)\,dx = \int_0^{\pi}\log(\sin x)\,dx \qquad (31)$$

which is a famous improper integral, it's $-\pi\log 2$ [5], and $\lim_{r\to 0} r\log(\sin\pi r) = 0$. Therefore

$$\int_{\ell_1}' f(z)dz + \int_{\ell_2}' f(z)dz = 2i\,(\log(\sin\pi r)+\log 2) + o(1) \qquad (32)$$

Now, all parts have been resolved. Seen from Fig. 2, one has

$$\int_{\ell}' f(z)dz = -(\int_{C_1}' f(z)dz + \int_{C_2}' f(z)dz + \int_{\ell_1}' f(z)dz + \int_{\ell_2}' f(z)dz)$$

$$= -[2(-i\,(\log r+2)) + 2i\,(\log(\sin\pi r)+\log 2)] + o(1)$$

$$= -2i\left(\log\frac{\sin\pi r}{r}+\log 2-2\right)+o(1) \qquad (33)$$

The value of this integration is independent of r; therefore, it equals its limit when r → 0. Now, one takes r → 0:

$$\int_{\ell}' f(z)dz = -2i\,(\log(2\pi)-2) \qquad (34)$$

By (8), (16),

$$\sum_{k=1}^{\infty} Res\,(f,k) = \frac{1}{4\pi i}\int_{\ell}' f(z)dz = \frac{1}{2\pi}\,(2-log(2\pi)) \qquad (35)$$

$$Res(f,k) = \frac{1}{\pi}\left[\left(k+\frac{1}{2}\right)log(1+\frac{1}{k})-1\right] \qquad (36)$$

One has:

$$\sum_{k=1}^{\infty}\left[\left(k+\frac{1}{2}\right)log\left(1+\frac{1}{k}\right)-1\right] = 1-\frac{\log(2\pi)}{2} \qquad (37)$$

Since one already proved that the equation above is equivalent to Stirling formula (by (6)), the proof is now complete.

3. Limitations and Prospects

This method only provides the initial version of the Stirling formula, which only considers integers (at most real numbers). In fact, this formula is applicable to all complex planes, and there are versions with error terms, which require the use of Γ's definition of infinite product of functions, rather than using the definition of integrals (because integral definitions cannot be extended to the entire complex plane). The final version of the Stirling formula can prove that Γ's definition of infinite product of a function is equivalent to the definition of integral. In fact, the author is attempting to directly derive the equivalence of these two definitions using methods like the contour integral and residue theorem in this article.

The Stirling formula has a wide range of applications in contemporary times, and its application in computer algorithms is most well-known because it leads to a more famous conclusion than itself, which is that the lower bound of the time complexity of comparison-based sorting algorithms is Ω (n log n) [5-7]. In addition, it even appears in solutions to some statistical mechanics problems [8-10], such as the extensibility paradox or the Gibbs' paradox [8]. It is evident that the Stirling formula, due to its simplicity and precision, still plays a new role in many new fields. It is believed that this long-standing formula will not easily disappear in the long river of scientific research in the future, at least its core idea has always inspired people to seek ways to simplify complexity in the process of researching any problem, when facing almost infinite numerical values.

4. Conclusion

To sum up, this approach using power series is quite useful for estimating and simplifying the calculation of contour integrals. Firstly, it allows for a quick determination that the integral along the periphery of the contour tends to zero. Then, during the calculation of the integral along a small arc, a situation arises where the singularity of the cotangent function aligns with the singularity of the logarithmic function. At this juncture, onwee can replace the cot z with 1/z to facilitate the estimation of the integral. In essence, power series can be a powerful tool for simplifying complex functions or dealing with singularities, ultimately providing a more straightforward means to tackle intricate problems. As mentioned earlier, the author is currently attempting to derive the equivalence of two definitions of Γ function using the method presented in this paper, and similar techniques are also used.

References

[1] Euler L. Methodus aequationes differentiales altiorum graduum integrandi ulterius promota. Novi Commentarii Academiae Scientiarum Petropolitanae, 1753: 3-35.

[2] Wästlund J. An elementary proof of the Wallis product formula for pi. The American Mathematical Monthly, 2007, 114(10): 914-917.

[3] Fairbanks L D. Notes on An Approach to Apery's Constant. arxiv preprint arxiv:2206.11256, 2022.

[4] Lairez D. Plea for the use of the exact Stirling formula in statistical mechanics. SciPost Physics Lecture Notes, 2023: 076.

[5] Cormen T H, Leiserson C E, Rivest R L, et al. Introduction to algorithms. MIT press, 2022.

[6] Martin D, Ahlfors L V. Complex analysis. New York: McGraw-Hill, 1966.

[7] Smolík J. An Elementary Proof of Stirling's Formula. arxiv preprint arxiv:2310.04872, 2023.

[8] Aycock A. Euler's First Proof of Stirling's Formula. arxiv preprint arxiv:2308.16313, 2023.

[9] Dalvit D A R, Frastai J, Lawrie I. Problems on statistical mechanics. CRC Press, 1999.

[10] Problems and solutions on thermodynamics and statistical mechanics. World Scientific, 2021.