Conditions for The Saturation of a Matrix Inequality

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Abstract. Matrix inequalities play a vital role in the study of linear algebra and information theory. In this paper, the author investigates the condition when a proven matrix inequality $r(\sum_{i=1}^{K} P_i \otimes S_i^T) \leq K \cdot r(\sum_{i=1}^{K} P_i \otimes S_i)$ is saturated, and Pi's are linearly independent matrices of the same size, and Si's are also linearly independent matrices of the same size. The author constructs the conditions when Pi is a column vector or a 2×2 matrix, and Si is an arbitrary square matrix. The author finds a very potential result when investigating the column situation. The author also gets massive result in the 2×2 situation. In particular, the inequality is not saturated when Pi is a 2×2 matrix and K=2 or 3 and it could be saturated under the situation when K=4 if only if two certain matrices can enable the product of $\sum_{i=1}^{K} P_i \otimes S_i^T$ and $\sum_{i=1}^{K} P_i \otimes S_i$ form two established matrices. At the end of the essay, the author tries some more complex cases such as 2×3 matrices Pi when K=2.

Keywords: Matrix, Kronecker product, partial transpose, Schmidt rank.

1. Introduction

Qian et al. proved a matrix inequality and this is the origin of this research. The inequality characterizes the tradeoff among the ranks of three bipartite reduced density operators of a tripartite quantum state. This is an innovative fact compared with the well-known subadditivity and strong subadditivity inequalities in terms of von Neumann entropy. Linden and Winter proved prove a new inequality for the von Neumann entropy which is independent of strong subadditivity [1]. Cadney et al. investigated relations between the ranks of marginals of multipartite quantum states [2]. They showed that there exist inequalities constraining the possible distribution of ranks. Kurzynski et al. presented a method to derive Bell monogamy relations by connecting the complementarity principle with quantum non-locality [3]. Higuchi et al. proved a necessary and sufficient condition for a set of none-qubit mixed states to be the reduced states of a pure n-qubit state is that their smaller eigenvalues should satisfy polygon inequalities: each of them must be no greater than the sum of the others [4]. Qian et al. proved a condition when the conjecture in entanglement distillability problem holds [5]. Horodecki et al. identified five selected open problems in the theory of quantum information [6]. Chen investigated the separability problem of rank at most 4 which are PPT [7].

In this paper, the author studies the conditions when the inequality $r(\sum_{i=1}^{K} P_i \otimes S_i^T) \leq K \cdot r(\sum_{i=1}^{K} P_i \otimes S_i)$ is saturated. The author shows three sections discussing a portion of this problem. In the first section, the author discusses the cases when P_i and S_i are all linearly independent vectors. In this case, we find that the inequality is constantly saturated. In the second section, we discuss all cases when P_i are all linearly independent vectors and S_i are all linearly independent n×n matrices. The author finds out that the inequality is saturated if and only if $r(S_1) = r(S_2) = \cdots = r(S_K) = r\left(\begin{bmatrix} S_1 \\ S_2 \\ \dots \\ S_K \end{bmatrix}\right) = \left(\frac{1}{K}\right) \cdot r([S_1 \quad S_2 \quad \dots \quad S_K])$. In the third section, this paper discusses the cases when P_i are 2×2 linearly independent matrices and S_i are linearly independent square matrices. In this case, this paper finds that the inequality cannot be saturated when K=2 or 3. When K equals to 4, this paper finds that the inequality can be saturated if and only if there exists two invertible matrix T and W that enables TMW and TM^{γ}W forms two specific matrices. In addition, the author investigates the case when P_i are 2×3 linearly independent matrices and S_i are linearly independent square matrices when K=2 the inequality cannot be saturated [8]. In this paper, the author uses SVD decomposition, matrix inequalities, matrix multiplication and other methods to reach our aims and make many discoveries and innovations.

2. Preliminaries and Notations

In this section, we are going to present some proven formulas or relations about Kronecker Product that are essential in further development [9].

Condition 1. $(A\otimes B)\otimes C = A\otimes(B\otimes C)$
Condition 2. $(A + B)\otimes C = A\otimes C + B\otimes C$
Condition 3. For scalar a: $a\otimes A = A\otimes a = aA$
Condition 4. For scalars a and b, $aA\otimes bB = abA\otimes B$
Condition 5. $(A\otimes B)(C\otimes D) = AC\otimes BD$
Condition 6. $(A\otimes B)^T = A^T\otimes B^T, (A\otimes B)^H = A^H\otimes B^H$
Condition 7. For vectors a and b, $a^T\otimes b = ba^T = b\otimes a^T$
Condition 8. For partitioned matrices, $[A_1, A_2]\otimes \mathrm{B} = [A_1\otimes \mathrm{B}, A_2\otimes \mathrm{B}]$
Condition 9. For square nonsingular matrices A and B: $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$
Condition 10. For m×m matrix A and n×n matrix B: $|\mathrm{A}\otimes\mathrm{B}| = |\mathrm{A}|^n|\mathrm{B}|^m$
Condition 11. rank $(A\otimes B)$ = rank (A)*rank (B)

Here, we are going to present some proven matrix inequalities or relations that are essential in further development. First, it is known that the rank of a matrix is invariant up to the multiplication of an invertible matrix. We shall frequently use this fact in the following sections, and may not stress it always. Second, it is known that the rank of a submatrix of Q is always less than or equivalent to the rank of Q. Third, $rank(A + B) \leq rank(A) + rank(B)$. Next, we present some notations.

$$M = \sum_{i=1}^{K} P_i\otimes S_i \tag{1}$$

$$M^{\gamma} = \sum_{i=1}^{K} P_i^T\otimes S_i \tag{2}$$

Here the matrices P_1,..., P_k are linearly independent blocks of the same size, and the matrices S_1,..., S_k are also linearly independent blocks of the same size. In addition, in this particular case, the Schmidt rank of M and Mγ equals to K. For convenience, we denote $r(A) = rank(A)$ and the matrix Iq as a $q \times q$ identity matrix [10].

3. Results and Discussion

We present some special cases in which the following inequality is saturated.

$$r(\sum_{i=1}^{K} P_i\otimes S_i^T) \leq K \cdot r(\sum_{i=1}^{K} P_i\otimes S_i) \tag{3}$$

Here the matrices P_1,..., P_k are linearly independent blocks of the same size, and the matrices S_1,..., S_k are also linearly independent blocks of the same size.

In the following, we introduce different conditions under which the inequality in (1) is saturated. In Sec. 3.1, we present the cases which P_j and S_j are all linearly independent vectors. In Sec. 3.2, we present the cases which P_j are all linearly independent vectors and S_j are linearly independent square matrices. In Sec. 3.3, we present the cases which P_j are all linearly independent 2×2 matrices and S_j are linearly independent square matrices. In Sec. 3.4, we present the cases which P_j are all linearly independent 2×3 matrices and Sj are linearly independent square matrices and K=2.

3.1. Pi and Si are All Vectors

We will present some situations that the inequality (1) is saturated when P1, …, PK are column vectors and S1, …, SK are also column vectors.

Lemma 1. The inequality is constantly saturated when $P_1, \ldots, P_K$ are linearly independent column vectors, and $S_1, \ldots, S_K$ are also linearly independent column vectors.

Proof. According to our preliminaries, $a \otimes b^T = ab^T$ which a and b are vectors, we can rewrite M^γ as $P_1 S_1^T + \cdots + P_K S_K^T$. We can create a matrix S, which $S = [S_1 \ \ldots \ S_K]$, since $S_1 \ldots S_K$ are linearly independent vectors, we can use row exchange to ensure that the first K rows of matrix S form a K×K invertible matrix S'.

Assume $S_i^T = [s_{i1} \ \ldots \ s_{i\alpha}]$, which $\alpha \geq K$ is the number of columns. Then we can rewrite $P_1 S_1^T + \cdots + P_K S_K^T$ as:

$$[P_1 s_{11} \ \ldots \ P_1 s_{1\alpha}] + \cdots + [P_K s_{K1} \ \ldots \ P_K s_{K\alpha}] = [P_1 s_{11} + \cdots + P_K s_{K1} \quad \ldots \quad P_1 s_{1\alpha} + \cdots + P_K s_{K\alpha}] \quad (4)$$

Only considering the first K, we assume:

$$c_1(P_1 s_{11} + \cdots + P_K s_{K1}) + \cdots + c_K(P_1 s_{1K} + \cdots + P_K s_{KK}) = 0 \quad (5)$$

$$P_1(c_1 s_{11} + \cdots + c_K s_K) + \cdots + P_K(c_1 s_{K1} + \cdots + c_K s_{KK}) = 0 \quad (6)$$

Because P1, …, PK are linearly independent, we have:

$$c_1 s_{11} + \cdots + c_K s_{1K} = \cdots = c_1 s_{K1} + \cdots + c_K s_{KK} = 0 \quad (7)$$

$$c_1(s_{11} + \cdots + s_{K1}) + \cdots + c_K(s_{1K} + \cdots + s_{KK}) = 0 \quad (8)$$

Thus, we have:

$$\begin{bmatrix} S_1^T \\ \ldots \\ S_K^T \end{bmatrix} \begin{bmatrix} c_1 \\ \ldots \\ c_K \end{bmatrix} = 0 = S'^T \begin{bmatrix} c_1 \\ \ldots \\ c_K \end{bmatrix} \quad (9)$$

Because S' is invertible, if and only if $c_1 = \cdots = c_K = 0$ the assumption is valid. Thus at least the first K columns of the result of $P_1 S_1^T + P_2 S_2^T + \cdots + P_K S_K^T$ is linearly independent. As the result, the rank of $\sum_{i=1}^K P_i \otimes S_i^T \geq K$. Because the rank of $P_i \otimes S_i^T$ equals the rank of P_i times the rank of S_i^T which is 1 and $r(A+B) \leq r(A) + r(B)$, $r(\sum_{i=1}^K P_i \otimes S_i^T) \leq r(P_1 S_1^T) + \cdots + r(P_K S_K^T) = K$. We have the rank of $\sum_{i=1}^K P_i \otimes S_i^T$ is K. Because the result of $\sum_{i=1}^K P_i \otimes S_i$ is a scalar, the rank of $\sum_{i=1}^K P_i \otimes S_i$ is 1. Thus $K \cdot r(\sum_{i=1}^K P_i \otimes S_i)$ is K, which means the inequality is saturated.

Lemma 2. The inequality is saturated when $P_1, \ldots, P_K$ and $S_1, \ldots, S_K$ are linearly independent row vectors, respectively.

Proof. For linearly independent column vectors $P_1^T, \ldots, P_K^T$ and $S_1^T, \ldots, S_K^T$, according to Lemma 1: $r(\sum_{i=1}^K P_i^T \otimes S_i^{TT}) = r(K \cdot \sum_{i=1}^K P_i^T \otimes S_i^T)$.

Because $A^T \otimes B^T = (A \otimes B)^T$, we have $r(\sum_{i=1}^K (R_i \otimes S_i^T)^T) = K \cdot r(\sum_{i=1}^K (P_i \otimes S_i)^T)$. Because transpose does not affect the rank of a matrix, we have:

$$r(\textstyle\sum_{i=1}^K P_i \otimes S_i^T) = K \cdot r(\textstyle\sum_{i=1}^K P_i \otimes S_i) \quad (10)$$

3.2. Pi are Vectors and Si are Square Matrices

We will present some situations that the inequality (1) is saturated when P1, …, PK are column vectors and S1, …, SK are square matrices.

Theorem 1. When $P_1, P_2, \ldots, P_K$ are linearly independent vectors and $S_1, S_2, \ldots, S_K$ are linearly independent $n\times n$ matrices, the inequality (1) is saturated if and only if $r(S_1) = r(S_2) = \cdots = r(S_K) = r\left(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix}\right) = (1/K) \cdot r([S_1 \quad S_2 \quad \ldots \quad S_K])$.

Proof. Because $P_1, P_2, \ldots, P_K$ are linearly independent vectors, we can find an invertible matrix U that enables $UP_1=\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$, $UP_2=\begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$, …, $UP_K=\begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$. We can express the rank of M as $r((U\otimes 1)M)$

Which is $r(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix})$. Because $A^T \otimes B^T = (A\otimes B)^T$, we have $\sum_{i=1}^{K} P_i \otimes S_i^T = \sum_{i=1}^{K} (P_i \otimes S_i^T)^T = \sum_{i=1}^{K} S_i^T \otimes S_i$. Using the same method, we can express the rank of M^γ as $r([S_1 \quad S_2 \quad \ldots \quad S_K])$. When the inequality (3) is saturated, we get:

$$r([S_1 \quad S_2 \quad \ldots \quad S_K]) = K \cdot r\left(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix}\right) \tag{11}$$

Because for any integer j between 1 and K, we have $r(S_j) \leq r(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix})$. We can deduce that:

$$K \cdot r\left(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix}\right) \geq r(S_1) + r(S_2) + \cdots + r(S_K) \tag{12}$$

We find that inequality (3) can be saturated if and only if $r(S_1) = r(S_2) = \cdots = r(S_K) = r(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix})$. Because r (A+B+…+K)≤r(A)+r(B)+…+r(K) where A, B, …, K are matrices, we can deduce that:

$$r([S_1 \quad S_2 \quad \ldots \quad S_K]) \leq r(S_1) + r(S_2) + \cdots + r(S_K) \tag{13}$$

From equation (2), we get inequality (3) and (4) must be saturated, which means:

$$r([S_1 \quad S_2 \quad \ldots \quad S_K]) = r(S_1) + r(S_2) + \cdots + r(S_K) = K \cdot r\left(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix}\right) = K \cdot r(S_j) \tag{14}$$

So, we have $r\left(\begin{bmatrix} S_1 \\ S_2 \\ \cdots \\ S_K \end{bmatrix}\right) = r(S_1) = \cdots = r(S_K)$.

3.3. Pi and Si are All Square Matrices

We will present some situations that the inequality (1) is saturated when P1, …, PK are square matrices and S1, …, SK are square matrices.

Lemma 3. The inequality (1) cannot be saturated when K=2; P_1, P_2, …, P_K are linearly independent 2×2 matrices and S_1, S_2, …, S_K are linearly independent n×n matrices.

Proof. We can use linear combination of P_1 and P_2 that enables one of them have rank 1 and another have rank 2. We assume P_1 have rank 2. Then we can find two multiplicative invertible matrices U and V that enables $UP_1V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $UP_2V = \begin{bmatrix} a_0 & a_1 \\ 0 & 0 \end{bmatrix}$. We can rewrite inequality (1) as:

$$2 * r(\begin{bmatrix} S_1 + a_0 S_2 & a_1 S_2 \\ 0 & S_1 \end{bmatrix} = r\left(\begin{bmatrix} S_1 + a_0 S_2 & 0 \\ a_1 S_2 & S_1 \end{bmatrix}\right) \tag{15}$$

We can find two invertible matrices U2 and V2 that enables $U_2 S_1 V_2 = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}$. We assume $U_2 S_2 V_2 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$. When we premultiplication each side by U2 and post multiplication each side by V2 we get:

$$2 * r(\begin{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} + a_0 \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} & a_1 \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\ 0 & \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = r(\begin{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} + a_0 \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} & 0 \\ a_1 \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} & \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}) \tag{16}$$

After elimination, we get:

$$2 * r(\begin{bmatrix} \begin{bmatrix} I_d + a_0 S_{11} & a_0 S_{12} \\ a_0 S_{21} & a_0 S_{22} \end{bmatrix} & \begin{bmatrix} 0 & a_1 S_{12} \\ 0 & a_1 S_{22} \end{bmatrix} \\ 0 & \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = r(\begin{bmatrix} \begin{bmatrix} I_d + a_0 S_{11} & a_0 S_{12} \\ a_0 S_{21} & a_0 S_{22} \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & 0 \\ a_1 S_{21} & a_1 S_{22} \end{bmatrix} & \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{17}$$

Which is equivalent to

$$2 * r(\begin{bmatrix} \begin{bmatrix} I_d + a_0 S_{11} & 0 \\ a_0 S_{21} & 0 \end{bmatrix} & \begin{bmatrix} 0 & a_1 S_{12} \\ 0 & a_1 S_{22} \end{bmatrix} \\ 0 & \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = r(\begin{bmatrix} \begin{bmatrix} I_d + a_0 S_{11} & a_0 S_{12} \\ 0 & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & 0 \\ a_1 S_{21} & a_1 S_{22} \end{bmatrix} & \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{18}$$

After the following elimination, we found that the inequality is constantly equivalent and cannot be saturated.

Lemma 4. The inequality (1) cannot be saturated when K=3; P_1, P_2, …, P_K are linearly independent 2×2 matrices and S_1, S_2, …, S_K are linearly independent n×n matrices.

Proof. In reference [9], the author proved that the block matrix $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ of Schmidt rank three is locally equivalent to $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & w \cdot N_{11} \end{bmatrix}$ where w is a complex number. Here M_{jk} and N_{jk} are n×n matrices. Thus, we can express the rank of M as the rank of $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & w \cdot N_{11} \end{bmatrix}$, the rank of M^γ as the rank of $\begin{bmatrix} N_{11} & N_{21} \\ N_{12} & w \cdot N_{11} \end{bmatrix}$. Here we have two situations, 1 and 2.

Situation 1. We have $w = 0$. In this case, we can express the rank of M as the rank of $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{bmatrix}$; the rank of M^γ as the rank of $\begin{bmatrix} N_{11} & N_{21} \\ N_{12} & 0 \end{bmatrix}$. When inequality (1) is saturated, we get $r\left(\begin{bmatrix} N_{11} & N_{21} \\ N_{12} & 0 \end{bmatrix}\right) = 3 * r(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{bmatrix})$. Because $r\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{bmatrix}\right) \geq r\left(\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}\right) = r(N_{11}) + r(N_{21})$, we get:

$$r\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{bmatrix}\right) \geq r(N_{11}), r(N_{12}), r(N_{21}), r(N_{11}) + r(N_{21}) \tag{19}$$

We also have $r\left(\begin{bmatrix} N_{11} & N_{21} \\ N_{12} & 0 \end{bmatrix}\right) \leq r(N_{11}) + r(N_{12}) + r(N_{21})$. Thus we conclude that $2 * r(N_{11}) + r(N_{12}) + 2 * r(N_{21}) \leq 3 * r(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & 0 \end{bmatrix}] = r\left(\begin{bmatrix} N_{11} & N_{21} \\ N_{12} & 0 \end{bmatrix}\right) \leq r(N_{11}) + r(N_{12}) + r(N_{21})$. This inequality can be valid only if N_{11}, N_{21} are zero matrices. In this condition, we get $r(N_{12}) = 3 * r(N_{12})$, thus N_{12} must be a zero matrix. Case 1 cannot be saturated.

Situation 2. We have $w \neq 0$. We assume a n×n matrix $F = w^{-1} \cdot I_n$.

In this case, we can express the rank of M as $r(M \cdot \begin{bmatrix} I_n & 0 \\ 0 & F \end{bmatrix}) = r\left(\begin{bmatrix} N_{11} & N_{12} \cdot F \\ N_{21} & N_{11} \end{bmatrix}\right) = r\left(\begin{bmatrix} N_{11} & w^{-1} \cdot N_{12} \\ N_{21} & N_{11} \end{bmatrix}\right)$ and the rank of Mγ as $r(\begin{bmatrix} I_n & 0 \\ 0 & F \end{bmatrix} \cdot M^\gamma) = r\left(\begin{bmatrix} N_{11} & N_{21} \\ F \cdot N_{12} & N_{11} \end{bmatrix}\right) = r\left(\begin{bmatrix} F \cdot N_{12} & N_{11} \\ N_{11} & N_{21} \end{bmatrix}\right) = r\left(\begin{bmatrix} N_{11} & F \cdot N_{12} \\ N_{21} & N_{11} \end{bmatrix}\right) = r\left(\begin{bmatrix} N_{11} & w^{-1} \cdot N_{12} \\ N_{21} & N_{11} \end{bmatrix}\right)$. Thus, the rank of M is constantly equivalent to the rank of Mγ. The inequality (1) cannot be saturated in case 2. The inequality (1) cannot be saturated in all of these cases.

Lemma 5. When K=4; $P_1, P_2, \ldots, P_K$ are linearly independent 2×2 matrices and $S_1, S_2, \ldots, S_K$ are linearly independent n×n matrices, the inequality (1) is saturated if and only if there exists two multiplicative invertible matrix T and W that enables TMW forms $\begin{bmatrix} I_d & 0 & 0 & I_d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_d & 0 & 0 & I_d \end{bmatrix}$ and TM^γW forms $\begin{bmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix}$.

Proof. We assume $M = \begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix}, M^\gamma = \begin{bmatrix} M_0 & M_2 \\ M_1 & M_3 \end{bmatrix}$. When inequality (1) is saturated, we have $4 * r(\begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix}) = r(\begin{bmatrix} M_0 & M_2 \\ M_1 & M_3 \end{bmatrix})$. Because for $0 \leq j \leq 3$, we have $\left(\begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix}\right) \geq r(M_j)$; $r\left(\begin{bmatrix} M_0 & M_2 \\ M_1 & M_3 \end{bmatrix}\right) \leq \sum_{i=0}^{3} r(M_i)$. We have $\sum_{i=0}^{3} r(M_i) \leq 4 * r(\begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix}) \leq \sum_{i=0}^{3} r(M_i)$. Thus $4 * r\left(\begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix}\right) = \sum_{i=0}^{3} r(M_i)$. Which means $r(M_0) = r(M_1) = r(M_2) = r(M_3) = r(\begin{bmatrix} M_0 & M_2 \\ M_1 & M_3 \end{bmatrix})$.

We assume there exist two n×n invertible matrices U and V that enable UM0V forms a diagonal matrix which rank is d. Then we can express the rank of M as:

$$r\left(\begin{bmatrix}U & 0\\0 & I_n\end{bmatrix}\begin{bmatrix}M_0 & M_1\\M_2 & M_3\end{bmatrix}\begin{bmatrix}V & 0\\0 & I_n\end{bmatrix}\right)=r\left(\begin{bmatrix}\begin{bmatrix}I_d & 0\\0 & 0\end{bmatrix} & UM_1\\M_2V & M_3\end{bmatrix}\right) \tag{20}$$

We can rename M1 and M2 as the result of the previous multiplication UM1 and M2V. We assume $M_j=\begin{bmatrix}M_{j0} & M_{j1}\\M_{j2} & M_{j3}\end{bmatrix}$ where the first block row/ column has d rows/ columns and the second block row/ column has (n-d) rows/ columns. Then We can express the rank of M as:

$$r\left(\begin{bmatrix}I_d & 0 & M_{10} & M_{11}\\0 & 0 & M_{12} & M_{13}\\M_{20} & M_{21} & M_{30} & M_{31}\\M_{22} & M_{23} & M_{32} & M_{33}\end{bmatrix}\right)=d \tag{21}$$

For $\begin{bmatrix}I_d & 0 & M_{10} & M_{11}\\0 & 0 & M_{12} & M_{13}\end{bmatrix}$, if M12 and M13 are not zero matrices, then they can never be eliminated by using linear combination. Thus, its rank must be greater than d, which is contradictory to our conditions. Thus, M12 and M13 must be zero matrices. Analogously, we can also prove that M21 and M23 are zero matrices. We can express the rank of M as the rank of $\begin{bmatrix}I_d & 0 & M_{10} & M_{11}\\0 & 0 & 0 & 0\\M_{20} & 0 & M_{30} & M_{31}\\M_{22} & 0 & M_{32} & M_{33}\end{bmatrix}$.

We get $r(M^\gamma)=r\left(\begin{bmatrix}I_d & 0 & M_{20} & 0\\0 & 0 & M_{22} & 0\\M_{10} & M_{11} & M_{30} & M_{31}\\0 & 0 & M_{32} & M_{33}\end{bmatrix}\right)=4d.$

Because $r(M^\gamma)\le r\left(\begin{bmatrix}I_d\\M_{10}\end{bmatrix}\right)+r(M_{11})+r\left(\begin{bmatrix}M_{20} & 0\\M_{22} & 0\\M_{30} & M_{31}\\M_{32} & M_{33}\end{bmatrix}\right)\le d+r(M_{11})+2d$. We have $d\le r(M_{11})\le d$, which means $r(M_{11})=d$. Analogously, we can prove that $r(M_{22})=d$. Because block row three/ block column three has d rows/ d columns, we can eliminate the blocks in row three and columns three by M11 and M22. Then we have:

$$r\left(\begin{bmatrix}I_d & 0 & 0 & 0\\0 & 0 & M_{22} & 0\\0 & M_{11} & 0 & 0\\0 & 0 & 0 & M_{33}\end{bmatrix}\right)=4d \tag{22}$$

We can deduce that r (M33) =d=r (M3). Considering M, we can use linear combination that enables M30, M31, M32 form zero matrices. M20, M22, M10, M11 after this manipulation becomes M’20, M’22, M’10, M’11 now we can express its rank as

$$r\left(\begin{bmatrix}I_d & 0 & M'_{10} & M'_{11}\\0 & 0 & 0 & 0\\M'_{20} & 0 & 0 & 0\\M'_{22} & 0 & 0 & M_{33}\end{bmatrix}\right)=d \tag{23}$$

We get $r(\begin{bmatrix} M'_{10} & M'_{11} \\ 0 & 0 \\ 0 & 0 \\ 0 & M_{33} \end{bmatrix}) \leq r(M) = d$. Because $r(M_{33}) = d$, we can deduce that M'10 is a zero matrix. Analogously, we can prove that M'20 is a zero matrix. Because the rank of M is d, we can deduce that M11, M22, M33 is a multiple I_d. Thus, we can use Id to replace M11, M22, M33 without changing the rank of M and Mγ.

In conclusion, the inequality (1) is saturated if and only if there exists two multiplicative invertible matrix T and W that enables TMW forms $\begin{bmatrix} I_d & 0 & 0 & I_d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_d & 0 & 0 & I_d \end{bmatrix}$ and TM^γW forms $\begin{bmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix}$.

3.4. 2×3 Matrices and Square Matrices

Lemma 6. When K=2; P_1, P_2, …, P_K are linearly independent 2×3 matrices and S_1, S_2, …, S_K are linearly independent n×n matrices, the inequality 1 cannot be saturated.

Proof. There exist two invertible matrices that enables $UP_1V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $UP_2V = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Here a, b, c, d, e, f are all scalar. We can time $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by a certain coefficient and add it to $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ that enables $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ have rank 1. After above manipulation, we can express the rank of P_2 as the rank of $\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & f_1 \end{bmatrix}$. Note that one of c_1 and f_1 must be non-zero or this will become the situation when P_j are 2×2 linearly independent matrices. Here we have two situations, one is $f_1 \neq 0$, another is $f_1 = 0$.

Situation 1. $f_1 \neq 0$. We can time the second row by a certain coefficient and add it to the first row, time the first column by a certain coefficient and add it to the second column so that we can cancel out c_1. Thus we can express the rank of P_2 as the rank of $\begin{bmatrix} a_2 & b_2 & 0 \\ 0 & 0 & f_1 \end{bmatrix}$. Here we find that b_2 must be non-zero or the linear combination of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 & 0 \\ 0 & 0 & f_1 \end{bmatrix}$ will produce a matrix which have rank 1 which is equivalent to situation 2. We can time $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ by a certain coefficient and add it to $\begin{bmatrix} a_2 & b_2 & 0 \\ 0 & 0 & f_1 \end{bmatrix}$ that enables $\begin{bmatrix} a_2 & b_2 & 0 \\ 0 & 0 & f_1 \end{bmatrix}$ forms $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & f_1 \end{bmatrix}$. When we time the third column by a certain coefficient, we can change f_1 into 1. Thus we can express the rank of M as the rank of $\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & S_1 & S_2 \end{bmatrix}$. We can also express the rank of M^γ as the rank of $\begin{bmatrix} S_1 & 0 \\ S_2 & S_1 \\ 0 & S_2 \end{bmatrix}$. From inequality (1), we have $2r(\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & S_1 & S_2 \end{bmatrix}) = r(\begin{bmatrix} S_1 & 0 \\ S_2 & S_1 \\ 0 & S_2 \end{bmatrix})$. Because $2r\left(\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & S_1 & S_2 \end{bmatrix}\right) \geq r\begin{pmatrix} S_2 & 0 \\ S_1 & S_2 \end{pmatrix} + r(S_1)$, we have $r(\begin{bmatrix} S_1 & 0 \\ S_2 & S_1 \\ 0 & S_2 \end{bmatrix}) \leq r\begin{pmatrix} S_2 & 0 \\ S_1 & S_2 \end{pmatrix} + r(S_1)$. Thus situation 1 cannot be saturated.

Situation 2. $f_1 = 0$. In this case, c_1 must be non-zero. We can use the third column to eliminate the first and second column of $\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \end{bmatrix}$ and time the third column by a certain coefficient then obtain $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. We can express the rank of M as the rank of $\begin{bmatrix} S_1 & 0 & S_2 \\ 0 & S_1 & 0 \end{bmatrix}$ and the rank of M^γ

as the rank of $\begin{bmatrix} S_1 & 0 \\ 0 & S_1 \\ S_2 & 0 \end{bmatrix}$. Thus we obtain $r(M^\gamma) \leq 2r(S_1) + r(S_2)$ and $r(M) \geq r(S_1) + r(S_2)$. We can deduce from inequality (1) that $2r(S_1) + 2r(S_2) \leq 2r(S_1) + r(S_2)$. Thus situation 2 cannot be saturated.

4. Conclusion

This paper investigated the condition by which inequality (1) is saturated. It is constantly saturated when P_j and S_j are linearly independent vectors, respectively. Next, we found the conditions when inequality (1) is saturated when P_j are linearly independent vectors or 2×2 matrices when K equals four and Sj are n×n matrices. Further, inequality (1) can never be saturated when K equals two or three. This paper also investigated the case when P_j is linearly independent 2×3 matrices. We think there is still a great potential to carry on for our Theorem 1. We also think that our research result can provide some equalities that are important in solving mathematical problems related to Schmidt rank, Kronecker product and partial transpose.

Still, this paper has some problems arising in this paper. In section three when $P_1, \ldots, P_k$ are square matrices and $S_1, \ldots, S_k$ are all square linearly independent matrices respectively, the author did not find out some general cases that enable the inequality (1) to be saturated. That is a great problem to carry on in the future. Besides, this paper may also try to extend the inequality to tripartite cases, and study the corresponding conditions for the saturation of inequality.

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