

The Transformative Role of Group Theory in Physics

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Abstract. This article investigates the significant impact of Group Theory within the field of physics, having a particular focus on its application in comprehending difficult physical systems and phenomena. This report centers on the application of Lie groups, rotational groups, the Poincaré and Lorentz groups in order to explain complicated features of physics. The report methodically discusses the role of the SU (2) group in Lie groups, the exploration of the rotational symmetries of the SO(3) group, and the significance of Lorentz transformations in special relativity with results demonstrating how Group Theory explains the nature of angular momentum in quantum mechanics and the limitations of the Schrödinger equation under Lorentz transformations. In addition, how the Poincaré group in special relativity is utilized. However, the scope of Group Theory's applications in physics is vast and multifaceted, making it challenging to encapsulate all its facets in a single report. This expansive field continues to evolve, promising further insights and innovations in the understanding of the physical universe.

Keywords: Lie groups; Poincaré group; SO (3); Schrödinger equation.

1. Introduction

Fundamentally, within the realm of mathematics, a group is defined as a collection of elements that has a binary operation and adheres to a set of predetermined characteristics [1]. The origins of this phenomenon may be traced back to the early 19th century, however, its profound implications in the field of physics were comprehensively recognized throughout the 20th century. The concept of symmetry has significant importance in the field of physics due to the presence of intrinsic symmetries in several physical systems [2]. In the field of physics, groups are used as a means to study and examine the symmetries that are manifested by particles, equations, and the fundamental laws that govern the world.

In the early 20th century, Noether's theorem, a significant contribution by Noether, unveiled a fundamental connection between symmetries and the preservation of physical quantities, commonly known as conservation laws. The author presented empirical evidence to substantiate the assertion that within a physical system, there exists a direct correlation between continuous symmetries and conserved quantities. The notable result has a profound impact on the advancement of modern physics, specifically within the domains of relativity theory and quantum mechanics [3]. The field of Group Theory offers a systematic and rigorous framework for the examination and categorization of symmetries, making it a valuable and effective tool in the realm of physics. Moreover, this mathematical concept possesses practical implications across diverse domains within the field of physics.

Notably, the utilization of groups aids in elucidating the process of quantifying angular momentum and comprehending the conduct of elementary particles like electrons and quarks. For instance, the orthogonal group in three dimensions, commonly denoted as SO (3), assumes a pivotal position in this context [4]. Within the discipline of electromagnetism, researchers investigate the fundamental principles that regulate electric and magnetic fields. This exploration has resulted in significant improvements in several technological domains, such as electronics and telecommunications [5].

Particle physics employs a methodology to substantiate the basic forces and constituents delineated by the standard model. This model incorporates the electromagnetic, weak, and strong nuclear forces, and it enables the classification of particles into separate groups based on their intrinsic properties [6].

Moreover, the field of condensed matter physics has included Group Theory as a fundamental tool for comprehending the dynamics of matter at both atomic and subatomic scales. The aforementioned

concept has played a pivotal role in elucidating several phenomena, including the process of crystal formation and the dynamics of electrons inside solid materials. The symmetries of crystal structures are described by crystallographic groups, which are highly dependent on the ideas of Group Theory.

The comprehension of particle symmetries has moreover facilitated the advancement of the notion of quarks and the Quantum Chromodynamics (QCD) hypothesis [7]. The categorization of hadrons, which are composite particles composed of quarks, into distinct groups like as baryons and mesons, is aided using Group Theory.

In brief, the discipline of Group Theory has had a significant influence on the realm of physics via the provision of a resilient and adaptable structure for comprehending and elucidating symmetries, transformations, as well as the basic forces and particles that constitute the cosmos. Group Theory has been crucial in molding the comprehension of the physical realm, spanning from Noether's theorem to the Standard Model and the investigation of condensed matter physics [8]. The applications of this phenomenon are wide-ranging, including the prediction of particle existence as well as the development of novel materials. Consequently, it remains a crucial instrument in the ongoing quest for scientific understanding and advancements within the field of physics. As this paper delves further into the topic matter, it will discuss particular instances and practical implementations that exemplify the revolutionary influence of Group Theory in the field of physics.

2. Methods

2.1. Lie Groups

One of the most utilized groups in physics is a Lie group which is a mathematical structure where the elements can be smoothly connected by paths and both the group operation and the inversion operation are smooth functions [9].

An instance of a Lie group is the group comprising 2×2 special unitary matrices. Characterized by complex number entries which can be called the SU (2) group short for Special Unitary group [10].

Definition 1. The SU (2):

$$SU(2) = \{ U \in \mathbb{C}^{2 \times 2} \mid U^\dagger U = I, \det(U) = 1 \} \quad (1)$$

The symbol U^\dagger denotes the conjugate transpose of matrix U , while I represent the 2×2 identity matrix. And the group must follow the following axioms [11].

Group Structure Axioms: Closure: For every pair of elements g, h in the group G , the product gh is also in G . Associativity: For every g, h, k in G , the equation $g(hk) = (gh)k$ holds. Identity: There is an element e in the group G such that for every element g in G , the equation $ge = eg = g$ is satisfied. Inverses: For each element g in G , there exists an element g^{-1} in G , such that $gg^{-1} = g^{-1}g = e$, where e is the identity element.

Differentiable Manifold Axioms: Smoothness: The group G is also a differentiable manifold, which means it has a structure that allows for the definition of smooth curves, surfaces, and other geometrical construct Compatibility of Group and Manifold Structures: The group operations (multiplication and inversion) are smooth maps. This means that the map $\mu: G \times G \rightarrow G$ defined by $\mu(g, h) = gh$ is smooth, and the inversion map $\iota: G \rightarrow G$ defined by $\iota(g) = g^{-1}$ is also smooth.

2.2. Rotational Group

At the most fundamental level, the rotational group encapsulates the symmetries of a sphere and more generally, the isotropy of space. Mathematically, a rotation is an orthogonal transformation that preserves the Euclidean distance and orientation in space.

Definition 2. Rotation: A linear transformation that preserves the norm of vectors and is orientation-preserving.

$$v' = R v \quad (2)$$

Where R is a 3×3 orthogonal matrix R with determinant $+1$.

Definition 3. Orthogonal Group $O(3)$: the group of all 3×3 orthogonal matrices, with real entries, that represent rotations and reflections in three-dimensional space [11].

$$O(3) = \{A \in GL(3, \mathbb{R}) : A^T A = I \text{ and } \det(A) = \pm 1\} \quad (3)$$

The general linear group $GL(3, \mathbb{R})$ consists of all 3×3 matrices with real entries that are invertible. The given object must adhere to the group axioms. An orthogonal matrix A is a matrix that satisfies the condition:

$$A^T A = A A^T = I \quad (4)$$

Here A^T is the transpose of A , and I is the identity matrix. The condition states that the inverse of an orthogonal matrix is its transpose.

Definition 4. The Special Orthogonal Group in three dimensions $SO(3)$ refers to the group of all three-dimensional rotations that preserve distances and orientations, preserving the origin and orientation consists of all 3×3 orthogonal matrices with a determinant of $+1$ [11].

$$SO(3) = \{R \in \mathbb{R}^{(3 \times 3)} \mid R^T R = I \text{ and } \det(R) = 1\} \quad (5)$$

Here, R^T represents the transpose of the matrix R , I is the 3×3 identity matrix, and $\det(R)$ is the determinant of R . It is a group under matrix multiplication; thus, it must follow the group axioms.

2.3. The Poincaré Group and the Lorentz Group

The Poincaré Group is distinguished among numerous groups due to its extensive use in several disciplines of physics, particularly in the fields of special relativity and quantum field theory. In order to comprehend the Poincaré Group, it is necessary to first establish the definition of the Lorentz group.

The Lorentz group, often represented as $O(1, 3)$ or L , is a mathematical group consisting of transformations that include rotations. It has significant importance in the realm of special relativity.

Definition 5. Lorentz Group is the set of all linear transformations that preserve the Minkowski metric.

$$O(1,3) = \{\Lambda \mid \Lambda^T \eta \Lambda = \eta\} \quad (6)$$

Where Λ is a 4×4 matrix representing a Lorentz transformation. η is the Minkowski metric, typically represented as a diagonal matrix $\eta = \text{diag}(-1, 1, 1, 1)$, which reflects the spacetime interval in special relativity. Λ^T is the transpose of Λ .

The Poincaré Group, which is a ten-parameter group, is formed by the combination of the Lorentz Group with translations in spacetime.

Definition 6. The Poincaré Group is the set of all affine transformations on Minkowski spacetime that preserve the spacetime interval between any two events.

$$P = \{(\Lambda, a) \mid \Lambda \in O(1,3), a \in \mathbb{R}^4\} \quad (7)$$

Λ represents a Lorentz transformation, a is a four-vector representing translations in space and time. And the general Poincaré transformation is (Λ, a)

Definition 7. A Poincaré transformation acts on a point in Minkowski spacetime x .

$$x' = \Lambda x + a \quad (8)$$

Where x' is the transformed point in spacetime.

3. Results and Discussion

3.1. The Association of Angular Momentum and $SO(3)$

Angular momentum in quantum mechanics is described by a set of operators that form a representation of the $SO(3)$ group which the operators are:

$$\widehat{L}_x = -i\hbar \left(-z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \right) \quad (9)$$

$$\widehat{L}_y = -i\hbar \left(-x \frac{\partial}{\partial z} + z \frac{\partial}{\partial x} \right) \quad (10)$$

$$\widehat{L}_z = -i\hbar \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) \quad (11)$$

Angular momentum operators in quantum mechanics $\widehat{L}_x, \widehat{L}_y, \widehat{L}_z$ are generators of the rotation group SO (3) as shown in definition 4. They correspond to infinitesimal rotations about the x, y, and z axes, respectively.

Lemma 1 the set of angular momentum operators $\{\widehat{L}_x, \widehat{L}_y, \widehat{L}_z\}$ satisfies the commutation relations of the SO (3).

Proof: By employing equations (9), (10), and (11) along with the commutation relations, the resultant outcome can be determined.

$$[\widehat{L}_x, \widehat{L}_y] = i\hbar \widehat{L}_z \quad (12)$$

$$[\widehat{L}_y, \widehat{L}_z] = i\hbar \widehat{L}_x \quad (13)$$

$$[\widehat{L}_z, \widehat{L}_x] = i\hbar \widehat{L}_y \quad (14)$$

Thus, it is consistent with the SO (3) algebra, and the Lie algebra structure is confirmed by performing analogous computations for the remaining commutators. The quantum states within a quantum mechanical system, particularly those that possess clearly defined angular momentum, serve as representations of the rotation group. The application of the angular momentum operators on these states is indicative of the fundamental rotational symmetry, as explained by the SO (3) group.

Example 1: Electron in a Hydrogen Atom: The eigenstates of an electron in a hydrogen atom are characterized by quantum numbers that derive from the representations of the SO (3) group. The quantum number l , also known as the orbital angular momentum quantum number, and m_l , which represents the magnetic quantum number are directly related to the eigenvalues of \widehat{L}^2 and \widehat{L}_z respectively (Fig 1).

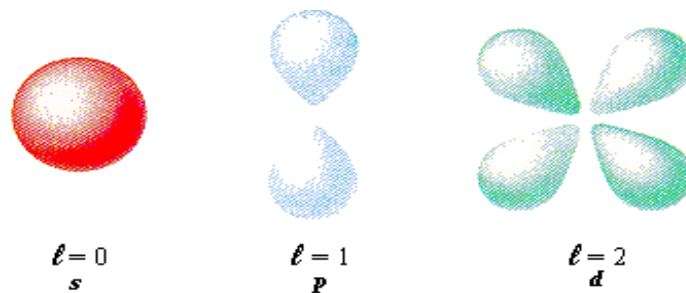


Fig 1. How the quantum number define the shape of the orbital [11].

3.2. Lorentz Group and the Schrödinger Equation

The Schrödinger equation, in its standard form, is not Lorentz invariant, meaning it does not hold the same form under Lorentz transformations which are central to the Lorentz group.

Theorem 1: The standard non-relativistic Schrödinger equation is not invariant under transformations of the Lorentz group.

Lemma 2: The non-relativistic Schrödinger equation does not maintain its form under Lorentz group transformations.

Proof: The Lorentz boost in the x-direction is represented as:

$$\Lambda_v^\mu(v) = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

Where $\gamma = 1/\sqrt{1-v^2/c^2}$ is the Lorentz factor? And the non-relativistic Schrödinger equation is given by:

$$i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \hat{H} \Psi(r, t) \quad (16)$$

Under a Lorentz transformation, time and space coordinates are mixed. However, the Schrödinger equation's different treatment of time and space violates this mixing, indicating that it is not Lorentz invariant.

Specifically, the time derivative would transform differently compared to the spatial derivatives under a Lorentz boost, leading to a different form of the equation in different inertial frames. Since the form of the Schrödinger equation changes under Lorentz transformations, it is not invariant under the Lorentz group.

3.3. Poincaré Group in Special Relativity

In special relativity, the Lorentz transformations, which are a part of the Poincaré group, preserve the spacetime interval between any two events in Minkowski spacetime from definition 5.

Theorem 2: Lorentz Transformation Preserves Spacetime Interval

Lemma 3: The Minkowski metric is invariant under Lorentz transformations, which implies $\Lambda_\alpha^\mu \Lambda_\beta^\nu \eta_{\mu\nu} = \eta_{\alpha\beta}$, where Λ is a Lorentz transformation matrix.

Proof of the Lemma: Using definition 5 and expanding the matrix multiplication:

$$\Lambda^T \eta \Lambda = \begin{pmatrix} \Lambda_0^0 & \Lambda_0^1 & \Lambda_0^2 & \Lambda_0^3 \\ \Lambda_1^0 & \Lambda_1^1 & \Lambda_1^2 & \Lambda_1^3 \\ \Lambda_2^0 & \Lambda_2^1 & \Lambda_2^2 & \Lambda_2^3 \\ \Lambda_3^0 & \Lambda_3^1 & \Lambda_3^2 & \Lambda_3^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda_0^0 & \Lambda_1^0 & \Lambda_2^0 & \Lambda_3^0 \\ \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\ \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \\ \Lambda_0^3 & \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3 \end{pmatrix} = \eta \quad (17)$$

This shows that the Lorentz transformation preserves the Minkowski metric.

Proof of the Theorem:

The spacetime interval ds^2 between two events in Minkowski spacetime is:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (18)$$

Under a Lorentz transformation from definition 6 the coordinates transform to,

$$x'^\mu = \Lambda_\nu^\mu x^\nu \quad (19)$$

Applying the lemma, we have

$$\eta_{\alpha\beta} \Lambda_\mu^\alpha \Lambda_\nu^\beta = \eta_{\mu\nu} \quad (20)$$

Therefore, the spacetime interval is invariant under Lorentz transformations, proving the theorem

$$ds'^2 = \eta_{\mu\nu} dx^\mu dx^\nu = ds^2 \quad (21)$$

The understanding of the properties of elementary particles, including their mass and spin, in the field of particle physics is achieved by examining their response to Poincaré transformations. An understanding of this concept is of crucial significance in the classification of particles and predicting of their interactions.

4. Conclusion

This report has demonstrated the extensive application of Group Theory in physics, particularly through the study of Lie groups, rotational groups, and the Poincaré and Lorentz groups. The methods employed included the analysis of the SU (2) group in Lie groups, the exploration of rotational symmetries via the SO (3) group, and the examination of Lorentz transformations in the context of special relativity. The results revealed the critical role of these groups in various physical phenomena. For instance, the association of angular momentum in quantum mechanics with the SO (3) group provided a deeper understanding of quantum states and their symmetries. Moreover, the analysis of the non-invariance of the Schrödinger equation under Lorentz transformations highlighted the nuanced interactions between quantum mechanics and relativity. These results highlight the significant impact of Group Theory in improving our comprehension of fundamental physics. However, the wide variety of applications of Group Theory in the field of physics is so broad that this report can only provide a limited overview of its extensive influence, showing an extensive field that is open to further investigation and advancement.

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