The Recurrence and Transience of Random Walks

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Abstract. This article reviews the research history and application fields of random walks and abstracts random walks into a time-homogeneous Markov chain to study their recurrent and transient properties. For one-dimensional and two-dimensional random walks, the likelihood of returning in n steps and the probability of returning for the first time in n steps of each state are first introduced, along with their relationship. Then the Stirling's formula is given, which is utilized to estimate the probability of returning in n steps, and the convergence and divergence of infinite series is used to prove that random walk in one dimension is recurrent. Random walk in two dimension has similar properties, which is a bit more complicated by using the relation between polynomials. Higher dimensional random walks need to consider special states first, and then generalize them to other states. Finally, this paper concluded that one-dimensional and two-dimensional random walks are recurrent, and random walks with dimensions higher than two are transient.

Keywords: Markov chain; Stirling's formula; estimation.

1. Introduction

In mathematics, a random walk is a statistical model made up of a sequence of random trends. An individual roaming around intoxicated creates random process recordings, which can be utilized to illustrate irregular fluctuation patterns. In1905, Karl Pearson made the initial proposal [1]. Random walks have many applications in various fields, such as in engineering and many scientific fields, including ecology, psychology, computer science, biology, physics, chemistry and economics [2]. In mathematics, we can estimate the value of π using a random walk of an individual-centered model. It can be used to model the paths of molecules as they travel through liquids or gases, the search paths of foraging animals, fluctuating stock prices and the financial well-being of gamblers. In these fields, random walks can be used to explain many observed phenomena and are therefore the fundamental statistical model for recording random activities.

On different spaces, random walks can be carried out. A variety of objects are frequently examined, including as surfaces, finitely generated groups, Lie groups, graphs, integer or real lines, vector spaces, and high-dimensional Riemannian manifolds [3, 4]. In the most basic scenario, time is discrete and the path of the random walk is a sequence of random variables indexed by natural numbers $X = \{X_1, X_2, \ldots, X_t, \ldots\}$.However, a random walk that moves at random intervals can also be defined in this way, in which case X must be defined for all times $t \in [0, +\infty)$. Considerable results have been achieved on the random walk problem. In 1977, Carazza summarized a sizable part of the history of the random walk problem [5, 6]. The recurrent nature of one- and two-dimensional random walks was first proved by George Pólya, which later became known as Pólya's random walk theorem [7, 8]. Carazza also proved that a random walk is transient if the dimension is not less than three [8]. Up to now, there are still many scholars studying this issue, Oriane Blondel, Marcelo R Hilário, Renato S dos Santos, Vladas Sidoravicius and Augusto Teixeira gives deeper insights into the high-dimensional random walk problem [9, 10].

In this paper, the random walks with low dimensions will be developed by the Markov chain to perform a transformation, while High-dimensional ones will also be mentioned to provide more inspiration.

2. Markov Chain

We introduce the idea of Markov chain to facilitate a better discussion of the likelihood of a particle executing a random walk and returning to the origin.

Definition 1. Stochastic process.

Consider $X = \{X_1, X_2, \ldots, X_t, \ldots\}$ is a sequence of random variables, and X_t represents a random variable at time t, where $t = 0,1,2, \ldots$ .The set of values of each random variable $X_t(t = 0,1,2, \ldots)$ is the same, which is called the state space and denoted by S. Random variables may be continuous or discrete. The random variables listed above are arranged in a stochastic process [10].

Definition 2. Markov property.

Assume that the random variable X_0 at time 0 follows a probability distribution $P(X_0) = p_0$,which is the initial state distribution. There is a conditional distribution $\mathrm{P}(X_t|X_{t-1})$ between the random variables X_t and X_{t-1} at a certain time $t > 1$, If X_t only depends on X_{t-1} and does not depend on the past random variables $\{X_0, X_1, \ldots, X_{t-1}\}$, this property is called Markov property [10], that is,

$$P(X_t|X_0, X_1, \ldots, X_{t-1}) = P(X_t|X_{t-1}) \tag{1}$$

The future is exclusively tied to the present, to put the Markov property in brief. This indicates that a process is presented that has both current and historical states, with the future state of the process depending only on the current state and being unrelated to the past state. We refer to this property as the Markov property. The most interesting point here is that some non-Markov processes can generate a Markov process by expanding the concepts of present and future states. We refer to this state as a second-order Markov process. Higher-order Markov processes can likewise be created by analogy [9].

Definition 3. Markov chain.

A Markov chain, or Markov process, is the random sequence $X = \{X_0, X_1, \ldots, X_t, \ldots\}$ with Markov properties. The conditional probability distribution $P(X_t|X_{t-1})$ is called the probability distribution of the Markov chain transition. The transition probability distribution determines the properties of the Markov chain [9]

The simplest Markov process is called a Markov chain, and it relates to the discrete exponential sets Markov process in particular. Through numerous trials, the probability distribution of each state may be calculated using the standard Markov chain, which primarily investigates the transition probability between the current and future states. This allows for the conversion of seemingly random events into overall orderly state changes.

Furthermore, If the transition probability distribution $P(X_t|X_{t-1})$ has nothing to do with t, that is,

$$P(X_{t+s}|X_{t-1+s}) = P(X_t|X_{t-1}), t = 1,2, \ldots; s = 1,2, \ldots \tag{2}$$

then the Markov chain is called a time-homogeneous Markov chain [10]. We have to deal with a time-homogeneous Markov chain as our random walk. Actually, there is an equal chance of traveling in any direction at any given time t. Assume S is the state space of a Markov chain, for any $i, j \in S$, we define,

$$f_{ij}^{(n)} = P(X_n = j, X_k \neq j, k = 1,2, \ldots, n-1|X_0 = i) \tag{3}$$

Where $f_{ij}^{(n)}$ is the probability that the Markov chain starts from state i at time 0 and reaches state j for the first time after n steps of movement, referred to as the first arrival probability. Let $f_{ij}^{(+\infty)} = P(X_n \neq j, n = 1,2 \ldots |X_0 = i)$ be the probability that the Markov chain starts from state i at time 0 and can never transition to state j. Let,

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} = P(\cup_{n=1}^{\infty} (X_n = j, X_k \neq j, k = 1,2, \ldots, n-1)|X_0 = i) \tag{4}$$

It is called the probability that the Markov chain starts from state i and finally reaches state j after a finite step transition. In particular, when $i = j$, f_{ii} represents the probability that the Markov chain starts from state i and finally returns to state i after a finite step transition.

Definition 4. Recurrent and transient.

Assume that S is the state space of the Markov chain and $i \in S$. If $f_{ii} = 1$, then the state i is said to be recurrent, and if $f_{ii} < 1$, the state i is said to be transient [10].

3. The Random Walk in One Dimension

We have introduced the concept of Markov chain and explained that random walk is a time-homogeneous Markov chain. Now we need to prove that the one-dimensional random walk is recurrent, that is to say, any state of the one-dimensional random walk is recurrent.

It is not difficult to find that each state of a one-dimensional random walk has the same recurrence and transience, so for the convenience of description, we abbreviate $f_{ii}^{(n)}$ as f_n.Calculating each f_n is a very complex process. In order to simplify the proof process, we introduce a new probability u_n,which refers to the one-dimensional random walk starting from state i and returning to i after n steps, not necessarily for the first time.

Lemma 1. We stipulate that $u_0 = 1, f_0 = 0$,then $u_n = \sum_{i=0}^{n} u_i f_{n-i}$ [10].

We multiply u_n by x_n, and we define,

$$U(x) = \sum_{n=0}^{\infty} u_n x^n \tag{5}$$

Do the same with f_n, then we have,

$$F(x) = \sum_{n=0}^{\infty} f_n x^n \tag{6}$$

Multiply $U(x)$ and $F(x)$ together. By Lemma 1 we find that,

$$U(x)F(x) = (\sum_{n=0}^{\infty} u_n x^n)(\sum_{n=0}^{\infty} f_n x^n) = (f_1)x + (f_2 + u_1 f_1)x^2 + \cdots + (\sum_{i=0}^{n} u_i f_{n-i})x^n + \cdots = u_1 x + u_2 x^2 + \cdots + u_n x^n + \cdots = U(x) - 1 \tag{7}$$

So we get the relation,

$$U(x) - 1 = U(x)F(x) \tag{8}$$

Which indicates that,

$$U(x) = \frac{1}{1 - F(x)} \tag{9}$$

Finally, plug in $x = 1$, we gain $U(x) = \frac{1}{1-F(x)}$, $\sum_{n=0}^{\infty} u_n = \frac{1}{1-\sum_{n=0}^{\infty} f_n}$. And by definition 4, the state is recurrent if $\sum_{n=0}^{\infty} f_n = 1$,which is equivalent to $\sum_{n=0}^{\infty} u_n = \infty$.

On the contrary, the state is transient if and only if $\sum_{n_0}^{\infty} u_n < \infty$,Now consider u_n. It is not difficult to find that when n is equal to an even number, u_n is 0. In fact, there are only two opposite transfer directions in a one-dimensional random walk, and in order to ensure that the transfer starts from state i and finally returns to state i, the total number of steps we get in the two directions must be equal. We find that,

$$u_{2n} = C_{2n}^{n} \left(\frac{1}{2}\right)^{2n} \tag{10}$$

The sum $\sum_{n=1}^{\infty} C_{2n}^{n} \left(\frac{1}{2}\right)^{2n}$ cannot be calculated straightforwardly. We need to make an estimate of this infinite series.

Lemma 2. Stirling's formula.

$$lim_{n \to +\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}} = 1 \tag{11}$$

Lemma 3. $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is an in infinite series. For $p \in (0,1]$, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges. For $p \in (1,\infty)$, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges [9].

Theorem 1. One-dimensional random walk is recurrent.

Proof. From Lemma 2, u_{2n} has the following estimation.

$$u_{2n} \sim \frac{\sqrt{2\pi 2n}(2n)^{2n}e^{-2n}}{\sqrt{2\pi n}n^n e^{-n}\sqrt{2\pi n}n^n e^{-n}} \cdot \left(\frac{1}{2}\right)^{2n} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \tag{12}$$

Therefore, $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ have the same convergence and divergence properties, and then according to Lemma 2. We conclude that $\sum_{n=0}^{\infty} u_n = \infty$, thus one-dimensional random walk is recurrent.

4. The Random Walk in Two Dimensions

A one-dimensional random walk can be considered on a number axis, where each state is an integer. Similarly, two-dimensional random walk can be considered in two-dimensional space. Each state corresponds to a coordinate. We use k_1,k_2,k_3 and k_4 to represent the numbers of stepup, stepdown, stepleft and stepright respectively. Suppose,

$$k_1 + k_2 + k_3 + k_4 = 2n, k_1 = k_2 = k, k_3 = k_4 = n - k \tag{13}$$

such that we could return back to origin point. Define f_n and u_n like before, we get that,

$$u_{2n} = \sum_{k=0}^{n} \frac{(2n)!}{k!\,k!\,(n-k)!\,(n-k)!}\left(\frac{1}{4}\right)^{2n} = \left(\frac{1}{4}\right)^{2n}\left(\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\frac{n!}{k!\,(n-k)!}\right)\frac{(2n)!}{n!\,n!}$$

$$= \left(\frac{1}{4}\right)^{2n}\left[\sum_{k=0}^{n}(C_n^k)^2\right]C_{2n}^n \tag{14}$$

Because of binomial expansion, we have the equation,

$$(1+x)^{a+b} = \sum_{n=0}^{a+b} C_{a+b}^n\, x^n \tag{15}$$

Simultaneously,

$$(1+x)^a = \sum_{n=0}^{a} C_a^n\, x^n \tag{16}$$

$$(1+x)^b = \sum_{n=0}^{b} C_b^n\, x^n \tag{17}$$

Multiply $(1+x)^a$ and $(1+x)^b$ together, we obtain,

$$(1+x)^{a+b} = \sum_{n=0}^{a+b}\sum_{i=0}^{n} C_a^i\, C_b^{n-i} x^n \tag{18}$$

Therefore,

$$C_{a+b}^n = \sum_{i=0}^{n} C_a^i\, C_b^{n-i} \tag{19}$$

Now we let $a = b = n$,finally we get that,

$$C_{2n}^n = \sum_{k=0}^{\infty}(C_n^k)^2 \tag{20}$$

Theorem 2. Two-dimensional random walk is recurrent

Proof. According to (20), $u_{2n} = \left(\frac{1}{4}\right)^{2n}\left[\sum_{k=0}^{n}(C_n^k)^2\right]C_{2n}^n = \left(\frac{1}{4}\right)^{2n} C_{2n}^n C_{2n}^n = \left[\left(\frac{1}{2}\right)^{2n} C_{2n}^n\right]^2$, by lemma 2, u_{2n} has the following estimation $u_{2n} \sim \frac{1}{\pi}\frac{1}{n}$. So $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ have the same convergence and divergence properties. By lemma 3, we conclude that $\sum_{n=0}^{\infty} u_n = \infty$, thus two-dimensional random walk is recurrent.

5. Random Walks in Higher Dimensions

Suppose a particle performs a symmetric random walk on an d-dimensional grid$(d > 2)$. Assume the origin is recurrent. If it starts from the origin, it will return to the origin with probability 1. Once it returns, the whole process will start again due to Markov properties. We find that it will come back with probability 1, so it will return to the origin infinite times. And if the origin is transient, then every time it returns to the origin and leaves again, it will never return with a certain positive probability, so its final number of returns is a geometric distribution with limited expectations. This shows that the origin is always recurrent if and only if the expectation of the number of times the particle returns to the origin after starting from the origin is infinite. It is always assumed below that the particles start from the origin. A list of indicative functions I_n is defined [10], it equals 1 if the particle returns to the origin at n steps, otherwise it is 0, then the expectation of the number of returns is,

$$E[\sum_{n=0}^{\infty} I_n] = \sum_{n=0}^{\infty} P(I_n = 1) \tag{21}$$

the order is changed because of the monotonic convergence theorem. Then just look at whether this series diverges or converges for different d. It is obvious that $I_n = 1$ can only be achieved if and only if is an even number. Next, we only consider $u_{2n} = P(I_{2n} = 1)$.

Theorem 3. For $d > 2$,the d-dimensional random walks are transient.

First consider the case $n = dm$, which means that n is the multiple of d.

$$u_{2n} = \frac{1}{(2d)^{2n}} \sum_{i_j \ge 0, \sum_{j=1}^{d} i_j = n} \frac{(2n)!}{\left(\prod_{j=1}^{d} i_j!\right)^2} = \frac{1}{(2d)^{2n}} \frac{(2n)!}{n!n!} \sum_{i_j \ge 0, \sum_{j=1}^{d} i_j = n} \frac{n!}{\prod_{j=1}^{d} i_j!} \frac{n!}{\prod_{j=1}^{d} i_j!} \le \frac{1}{(2d)^{2n}} \frac{(2n)!}{n!n!} \sum_{i_j \ge 0, \sum_{j=1}^{d} i_j = n} \frac{n!}{\prod_{j=1}^{d} i_j!} \frac{(dm)!}{(m!)^d} = \frac{1}{2^{2n} d^n} \frac{(dm)!}{(m!)^d} \frac{(2n)!}{n!n!} \tag{22}$$

By Lemma 2, u_{2n} has the following estimation.

$$u_{2n} = \frac{1}{2^{2n} d^n} \frac{(dm)!}{(m!)^d} \frac{(2n)!}{n!n!} \sim \frac{C_d}{n^{\frac{d}{2}}} \tag{23}$$

Where C_d is a constant. By Lemma 3, we know that $\sum_{m=0}^{\infty} u_{2dm}$ converges. For $n \ne dm$, if $n = dm - 1$, we have $u_{2dm} = P(I_{2dm} = 1) \ge P\left(I_{2dm} = 1, I_{2(dm-1)} = 1\right) = P\left(I_{2(dm-1)} = 1\right) P\left(I_{2dm} = 1 \middle| I_{2(dm-1)} = 1\right) = P\left(I_{2(dm-1)} = 1\right) \frac{1}{2d} = \frac{1}{2d} u_{2(dm-1)}$. So $u_{2(dm-1)} \le 2d \cdot u_{2dm}$. Which means that $\sum_{m=1}^{\infty} u_{2(dm-1)}$ also converges. Similarly we can prove that $\sum_{m=1}^{\infty} u_{2(dm-p)}$ all converge for $p = 2,3, \dots, d-1$. We conclude that $\sum_{n=0}^{\infty} u_n$ converge for $d > 2$, thus d-dimensional random walks are transient.

6. Conclusion

In this paper, the definition of random walk is given. It is shown that a stochastic process with Markov properties is called a Markov chain, and the theorem that a random walk is a time-homogeneous Markov chain is simply proved. The two properties, recurrent and transient properties are introduced and a new probability u_n is defined to establish its connection with f_n. With the help of this connection, the equivalence conditions of recurrent and transient random walks are deduced. In the proof process of the recurrence of one-dimensional random walk, two important lemmas, namely Stirling's formula and the convergence and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ series, are explained, which then are used to estimate the series$\sum_{n=0}^{\infty} u_n$, and finally proved that the one-dimensional random walk is recurrent. In addition, the equation $C_{2n}^{n} = \sum_{k=0}^{n} (C_n^k)^2$is derived using the relationship between polynomials. Then $\sum_{n=0}^{\infty} u_n$ is estimated again using a similar method as before, which shows that the two-dimensional random walk is also a regular return. For high-dimensional random walks, we first analyze the special terms in the series $\sum_{m=0}^{\infty} u_{2dm}$, and then generalize to other terms.

It is proved that the series $\sum_{n=0}^{\infty} u_n$ is convergent, and then shows that high-dimensional random walk is transient.

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