

Research on Locating and Searching for Missing Submersibles Based on Differential Equations and SAR-A* Algorithm

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Abstract. In recent years, the application of submersibles has become increasingly widespread. The current focus of research is on real-time and accurate positioning of submersibles to cope with potential emergencies. This paper aims to establish a trajectory model for faulty submersibles. Firstly, a differential equation model for submersible dynamics is established, numerical solutions are obtained and fitted to determine the exponential relationship between velocity and time, resulting in a theoretical formula for submersible displacement. Subsequently, based on the trajectory, a probability model for the presence of targets is established using a two-dimensional Gaussian distribution. The SAR-A* algorithm is employed for regional search, combined with a greedy algorithm to obtain initial deployment points for search and rescue, thus deriving the search and rescue trajectory. This study provides theoretical support for locating and rescuing missing submersibles and is expected to play an important role in future submersible applications.

Keywords: Differential Equations, Submersible Positioning, SAR-A*.

1. Introduction

With the continuous progress of technology, there are more underwater positioning methods. Submersible positioning methods mainly include TOA^[1] and TDOA^[2], etc. These methods are based on sensors and cannot achieve real-time prediction of the lost submersible. However, it is necessary to predict the position of submersible in real time, so this paper will give a prediction model of the lost submersible according to the dynamic relationship of submersible.

The application of AUV has been increasingly widespread^[3]. To enable AUVs to efficiently complete tasks and minimize movement distances, research on path planning methods is particularly important^[4]. The main methods for path planning include the Dijkstra algorithm^[5], the A* algorithm^[6], and the artificial potential field method^[7]. Unlike the widely studied target search strategy^[8], this paper focuses on completely covering the search area, prioritizing the search of regions with higher probabilities of target presence, ensuring efficient completion of search and rescue tasks.

2. Establishment of differential equation of submersible dynamics

2.1. Coordinate Establishment and Basic Assumptions

As shown in Figure 1, we first establish a three-dimensional coordinate system with the Earth as reference, denoted as $ox - oy - oz$. The horizontal velocity of the submersible relative to the Earth coordinate system is v . The components of velocity v along the $x - axis$ and $y - axis$ are v_x and v_y respectively. The velocity of the submersible along the $z - axis$ is v_z . The velocity of the ocean current relative to the Earth coordinate system is v_o , with its components along the $x - axis$ and $y - axis$ being v_{ox} and v_{oy} respectively.

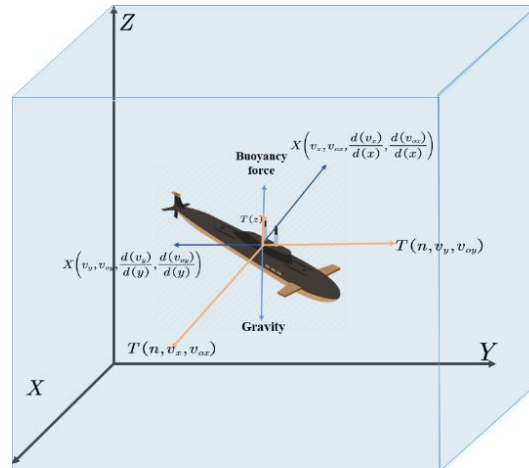


Figure1: Force Analysis of the Submersible

Taking into account disturbances in the vertical direction caused by factors such as terrain and density within the seawater, this paper assumes that the velocity of the submersible in the vertical direction follows a uniform distribution of $[-0.001, 0.001]$. Additionally, this paper does not consider vertical movement of ocean currents, treating them as stable flows where disturbances in the vertical direction of the submersible do not affect the current velocity. Regarding the submersible, this paper considers it as a point mass and assumes that it immediately loses power upon becoming lost.

2.2. Dynamics Differential Equations

Since the analysis for the x and y directions is similar, we will primarily focus on the analysis of the x – *direction*. According to the principles of rigid body motion, the motion equation for the submersible in the x – *direction* is given by:

$$m \frac{dv_x}{dt} = T(n, v_x, v_{ox}) + X(v_x, v_{ox}, \frac{dv_x}{dt}, \frac{dv_{ox}}{dt}) \quad (1)$$

Where m is the mass of the submersible, T is the thrust generated by the propulsion system in the x – *axis* direction, and X represents the hydrodynamic component of the submersible in the x – *axis* direction.

Applying the principle of relativity and concepts from fluid mechanics, we decompose the second term on the right-hand side of the equation [9]. Then, substituting the result into equation (1), we obtain:

$$(m + m_1) \frac{dv_x}{dt} = X_1(v_x - v_{ox}) - m_1 \frac{dv_{ox}}{dt} \quad (2)$$

Where m_1 represents the added mass of the submersible, and X_1 represents the resistance of the submersible.

Considering the variation range of Reynolds numbers during low-speed navigation of the submersible and the possibility of changes in the drag coefficient of general objects within a certain range of Reynolds numbers, this paper adopts a cubic form:

$$X_1(v_x) = X_u v_x + X_{uu} v_x^2 + X_{uuu} v_x^3 \quad (3)$$

The final dynamic differential equation for the submersible is obtained as follows:

$$(m + m_1) \frac{dv_x}{dt} = X_u (v_x - v_{ox}) + X_{uu} (v_x - v_{ox})^2 + X_{uuu} (v_x - v_{ox})^3 - m_1 \frac{dv_{ox}}{dt} \quad (4)$$

2.3. Establishment of the Submersible Displacement Equation

The simulated object of this paper is a 3,000-ton submersible, the main parameters of which are derived from the literature^[10], and the selection of additional mass and hydrodynamic coefficient is mainly referenced in the literature^[11-12]. The parameters selected by the simulated submersible are shown in Table 1.

Table 1: Basic Data and Hydrodynamic Coefficients of the Submersible

Parameter	Unit	Value
Mass	<i>kg</i>	$3 * 10^5$
Added Mass	<i>kg</i>	$1.178 * 10^4$
X_u	<i>kg/s</i>	$9.317 * 10^2$
X_{uu}	<i>kg/s</i>	$9.317 * 10^2$
X_{uuu}	<i>kg/m</i>	$-1.577 * 10^3$

After obtaining the parameters, numerical solutions of the differential equations will be computed using MATLAB. Based on the numerical solutions of velocity, changes in displacement will be derived. Figure 2 shows the solutions for the x direction with stable ocean current velocity component $v = 2kn$, and submersible initial speed $v_0 = 4kn, 1.5kn$. Figure 3 illustrates the solutions for the x direction with stable ocean current velocity component $v = 3kn$, and submersible speed $v_0 = 5kn, 2kn$.

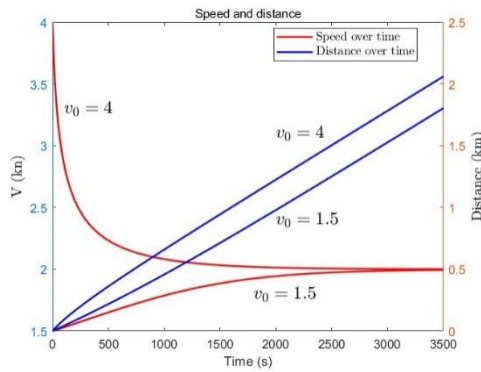


Figure 2: Speed and distance for $v = 2kn$

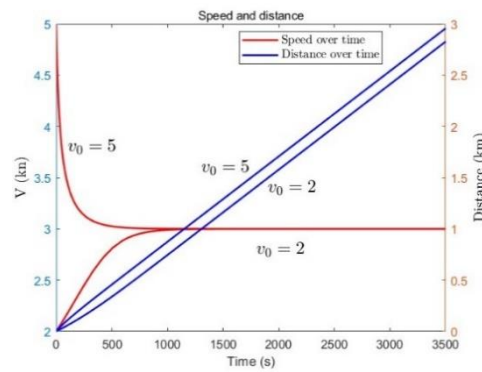


Figure 3: Speed and distance for $v = 3kn$

The results show that when the submersible loses power, regardless of its speed at the time of loss, the final speed of the submersible will tend toward the velocity of the ocean current, indicating that the submersible will ultimately drift with the current. Nonlinear regression analysis was conducted on the submersible speed in this study, revealing that the fitting results of exponential functions are relatively accurate, with coefficients of determination R^2 all exceeding 0.9. The following velocity formulas were obtained:

$$V_x(t) = v_{ox} + (v_x - v_{ox})e^{-K_u t} \quad (5)$$

Let's define k_u as the "drift index", whose value can be obtained through least squares fitting. By integrating equation (5) with respect to time t , we obtain the function relationship between the displacement distance of the submersible and time.

$$\varepsilon_x(T) = v_{ox} T - \frac{(v_{ox} - v_x)}{K_u} (e^{-K_u T} - 1) \quad (6)$$

2.4. Simulation and Modeling

According to the obtained displacement function, this paper predicts the position of the submersible. Assuming the initial speed of the submersible $v_0 = 5kn$, ocean current velocity $v = 3.5kn$, the angle between the ship speed and the ocean current velocity is $\pi/5$, and the angle between the ocean current velocities is $\pi/6$. The simulation time is 3500s. After obtaining the numerical solution of the differential equation, we fit v_x and v_y separately to obtain $k_u x = 0.01062$ and $k_u y = 0.002769$. The results are shown in Figures 4 and 5, with goodness of fit being 0.987 and 0.934 respectively. The analytical solution of velocity matches well with the numerical solution.

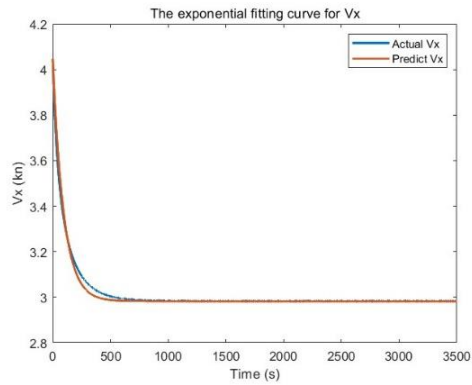


Figure 4: Theoretical Velocity and Actual Velocity in the x

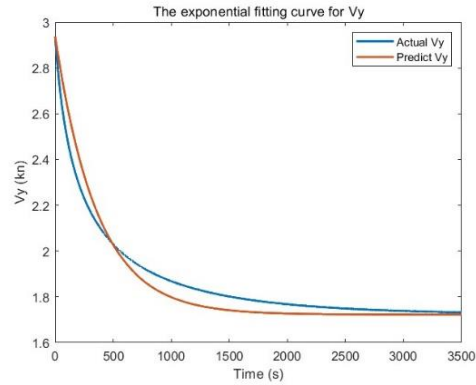


Figure 5: Theoretical Velocity and Actual Velocity in the y

Substituting $k_u x$ and $k_u y$ into the equation (6), we obtained the theoretical trajectory and predicted trajectory as shown in Figures 6 and 7, respectively. It can be observed that the predicted trajectory curve is very close to the actual trajectory curve, indicating that the function relationship between displacement and time is applicable. Furthermore, it is noticed that in the early stages of submersible loss, the submersible displacement may slightly deviate from the predetermined trajectory, reaching the ocean current velocity over time and approximately moving in a straight line.

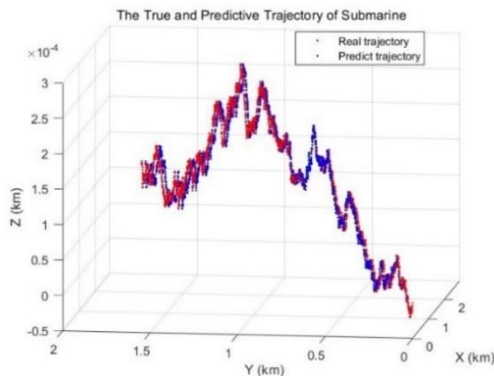


Figure 6: Theoretical Position and Predicted Position

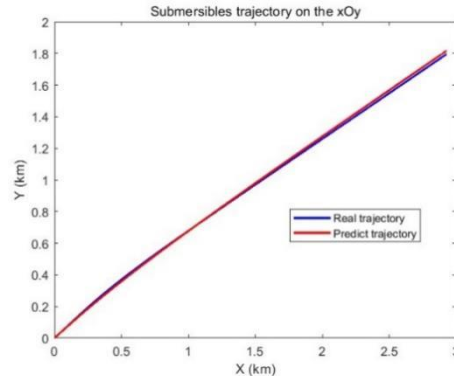


Figure 7: Theoretical Position and Predicted Position in Plane

3. Search and Rescue of Missing Submersibles Based on SAR-A* Algorithm

3.1. Basic Principles of the SAR-A* Algorithm

The A* algorithm is a heuristic algorithm commonly used to reduce redundant paths for robots. The traditional A* algorithm is used to find the shortest path from a starting point to an endpoint while avoiding obstacles based on a known static map ^[13]. The focus of the A* algorithm is to find the next iteration point.

The SAR-A* algorithm used in this paper ^[14] is designed for complete coverage of the search area. It is based on the A* algorithm, which divides the search area into grids and achieves full coverage by traversing all grids. It requires the establishment of a suitable probability model and construction of a heuristic function. The next iteration point is determined through the heuristic function to find the complete search path. Figure 8 illustrates the flowchart of the SAR-A* planning process.

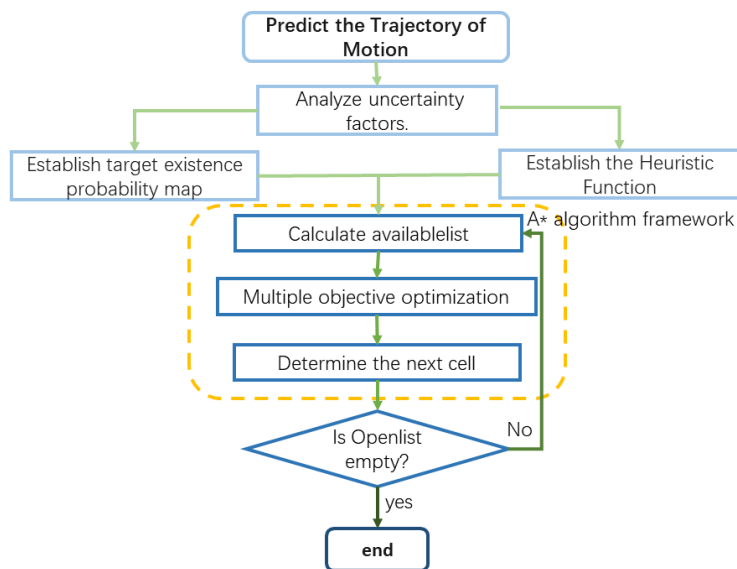


Figure 8: Flowchart of the SAR-A* Algorithm

3.2. Gridification of the Search Area and Construction of the Probability Model

Using the dynamic model derived from the differential equations, the approximate position of the submersible can be predicted. The search and rescue equipment in this paper is an AUV, primarily utilizing side-scan sonar for search operations. Considering various uncertainties in the early stages of emergency situations, it is assumed that all preparations are completed 30 minutes after the occurrence of the fault, and the AUV officially begins the rescue mission. Using the data from Section 2.4, after obtaining the approximate position of the missing submersible, a rectangular area centered around this position will be chosen as the search area, which will then be gridified. The result of gridification is shown in Figure 9.

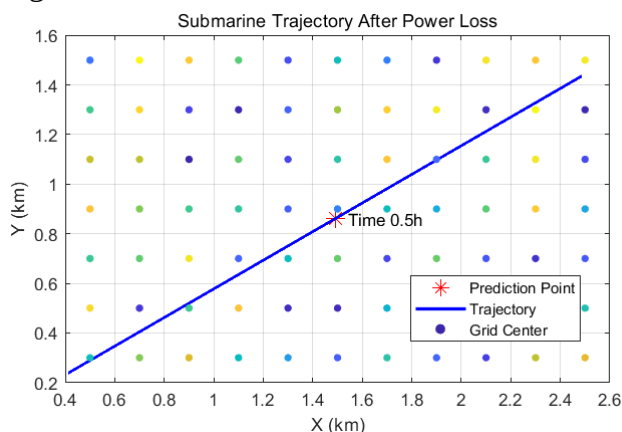


Figure 9: Grid diagram

Considering the complexity of ocean currents, the submersible will not strictly drift along the trajectory after becoming lost, but it will also not deviate significantly from the predetermined trajectory. Therefore, this paper adopts a bivariate Gaussian distribution to approximate the probability of the submersible's presence. The mean is set to (6, 4), and the covariance matrix of the Gaussian distribution is adjusted based on the data from Section 2.4, including angles and trajectories.

Finally, the probability density map is obtained as shown in Figure 10, and the probability distribution map is shown in Figure 11.

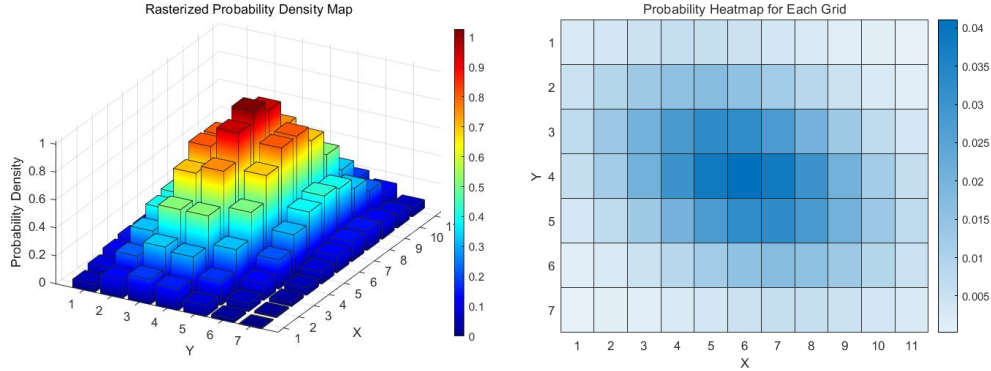


Figure 10: Rasterized Probability Density Map **Figure 11:** Probability Heatmap for Each Grid

3.3. Establishment of the Heuristic Function for SAR-A* Algorithm

In path planning problems, the choice of the next iteration point is the foundation of the entire search problem, and the heuristic function is crucial for determining the next iteration point. To avoid unnecessary distances, the AUV prioritizes selecting grids that are closer to the current cell and closer to potential target positions. In Figure 12 (a), when the rescue vessel is at point A, if points C and D have not been visited yet, point B should not be used as the next path point. Based on this, the objective function f_1 is formulated.

$$\begin{cases} \min f_1 = w_{d0} \frac{d(s_{now}, s_i)}{d_0} + w_{dh} \frac{d(s_h, s_i)}{d_h}, s_i \in S_{ava} \\ d_0 = \max d(s_j, s_1), s_j \in S_{all} \\ d_h = \max d(s_k, s_h), s_k \in S_{all} \\ w_{d0} + w_{dh} = 1 \end{cases} \quad (7)$$

Where $S_{ava} = S_{nei} \cap S_{open}$, S_{open} represents all unvisited points, S_{nei} represents the neighboring nodes of S_{now} , and when S_{nei} is empty, $S_{ava} = S_{all} \cap S_{open}$ represents the current position, S_i represents the selectable points in S_{ava} , s_h represents the point with the maximum probability, $d(s_{now}, s_i)$ and $d(s_h, s_i)$ represent the Manhattan distances between two points, d_0 represents the maximum distance from all points to the initial point, d_h represents the maximum distance between the point s_k containing the minimum value p_i and the point s_h containing the maximum value p_i , and w_{d0}, w_{dh} represent weights.

Considering that turning in the ocean can cause distortion in side-scan sonar^[15], the AUV needs to avoid turning as much as possible. In Figure 12 (b), when the last movement was from E to F, the next optimal choice point is G, and the secondary choice points are H and I. Based on this, the objective function f_2 is formulated.

$$\min f_2(s_i) = \frac{\theta(s_{par}, s_{now}, s_i)}{180}, s_i \in S_{ava} \quad (8)$$

Where $\theta(s_p, s_{now}, s_i)$ represents the angle between the next iteration point and the current iteration point, and the current iteration point and its parent node.

In order to quickly locate the missing submersible, it is necessary to prioritize searching for points with higher probabilities. In Figure 12 (c), when the rescue vessel is at point J, if point K has not been visited yet, points M and L should not be used as the next path points. Here, the objective function f_3 is formulated.

$$\max f_3(s_i) = \frac{p_l(s_i) - p_{l\min}}{p_{l\max} - p_{l\min}}, s_i \in S_{ava} \quad (9)$$

Where $p_{l\min}$ is the minimum probability, $p_{l\max}$ is the maximum probability, and $p_l(s_i)$ represents the probability of the next iteration point.

In summary, this paper adopts a weighted measure method to evaluate the next iteration point. It is easy to see that the ideal solution is $f_1^* = 1$, $f_2^* = 0$, $f_3^* = 1$. Therefore, the final heuristic function is:

$$\min H(s_i) = \sum_{m=1}^3 \eta_m (f_m(s_i) - f_m^*)^2, s_i \in S_{ava} \quad (10)$$

Where $H(S_i)$ is the final evaluation of the selectable cell, and η_m is the corresponding weight.

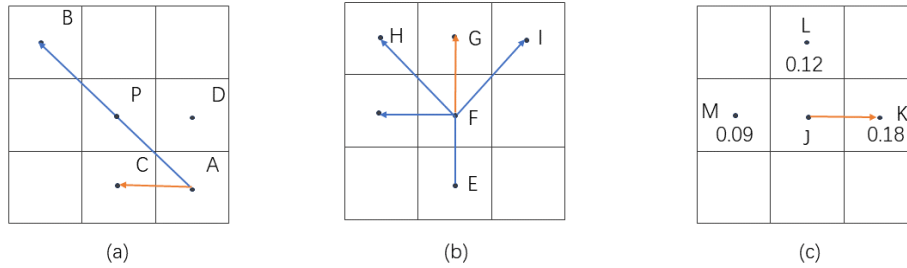


Figure 12: Criteria for selecting the next waypoint

3.4. Simulation Results

After multiple simulation results, it is observed that when the probability reaches 0.8, the search path will follow the outer region. Therefore, this paper assumes that when the cumulative probability of the search path reaches 0.8, the AUV successfully locates the missing submersible. To determine the initial search point, this paper first uses the greedy algorithm to traverse all possible points and selects the point that can reach the probability threshold in the shortest time as the initial point. It is calculated that when the initial deployment point is (6,3), the minimum number of steps to reach a cumulative probability of 0.8 is 36. Here, the trajectory plot is provided for the initial deployment point (6,3), as shown in Figure 13.

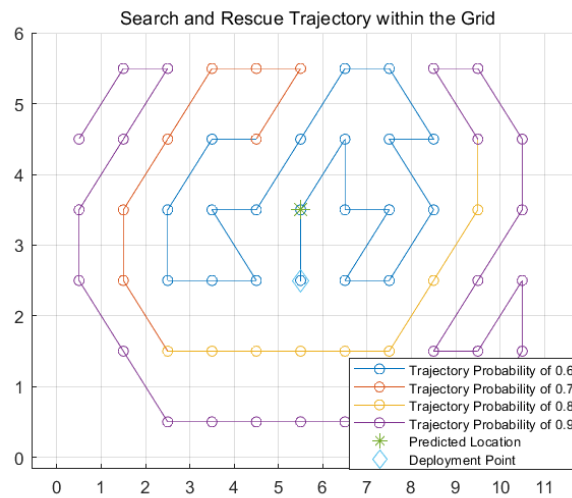


Figure 13: Search and Rescue Trajectory within the Grid

It can be observed that in the early stages of the search and rescue operation, the AUV first conducts a search around the predicted position of the submersible, gradually expanding outward. From the trajectory analysis, the search mainly focuses on areas with high probabilities and minimizes the number of turns, ensuring the effectiveness of the search and rescue operation. Additionally, there

are no situations similar to Figure12 (a), indicating that the search and rescue method did not encounter "stuck" issues, significantly improving the efficiency of the search and rescue operation.

4. Conclusion

This paper presents a differential equation model for the underwater dynamics of a submersible, used to predict the position trajectory of a faulty submersible. Subsequently, the paper selects an AUV equipped with side-scan sonar for underwater rescue operations and provides a method for selecting the initial deployment point. Simulation results using the SAR-A* algorithm are provided to illustrate the search and rescue route of the AUV. However, this paper has some limitations: it does not consider underwater terrain, marine life, and underwater obstacles, and some assumptions make the model significantly different from reality. Future work will focus on addressing these issues to improve the model.

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