Application of Functional Analysis in Solving River Crossing Problems under Dynamic Water Flow Conditions

Zhe Zhang *

Southeast University Nanjing, China

* Corresponding Author Email: 213212053@seu.edu.cn

Abstract. Considering the urgent need for efficient and safe river crossing strategies in natural disaster relief and outdoor sports, in-depth research on river crossing strategies based on mathematical modeling and optimization theory is of great significance for improving river crossing safety and efficiency and guiding practical operations in related fields. Aiming at the goal of "rushing across the Yangtze River", based on the variational method and functional analysis theory, the Hamilton-Lagrange multiplier method is used to solve the constrained functional extremum problem. By solving the Euler equation, a universal continuous-time optimal control model is established. Using dichotomy and numerical analysis, the optimal trajectory and direction control law of swimming speed is solved, and the optimal path scheme to reach the endpoint is proposed, which contributes to the efficiency and safety of river crossing and provides practical suggestions in related fields.

Keywords: Functional analysis, Extremum problem with constraints, optimal control theory, Dichotomy, Interpolation.

1. Introduction

Complex optimization problems in real life often need the support of mathematical theory. How to find the optimal solution under various constraints has been the focus of constant exploration in science and engineering. Especially in areas such as natural disaster response, sports, engineering planning, and environmental management, effective optimization strategies can significantly improve efficiency and safety. Taking the problem of "rushing across the Yangtze River" as an example, this study explores the optimal way to achieve the goal under given natural conditions and physical limitations. As one of the longest rivers in the world, the complex hydrological characteristics of the Yangtze River pose additional challenges for river crossing. The solution of such problems is not only of great significance for theoretical research but also has a profound impact on practical application scenarios such as river rescue and sports training. As a result, we take the swimmer's velocity direction θ(y) and the distribution of water velocity v(y) as the functional parameters to propose the optimal path scheme to reach the endpoint.

This research is carried out using dichotomy, numerical analysis, and the Hamilton-Lagrange multiplier method based on the variational method and functional analysis theory. This method was first introduced by Euler and Lagrange in the 18th century and has since been further developed and extended by numerous mathematicians. Now it is used to study constrained functional extremal problems, such as Minimal surface problems [1], constrained extremal problems for nonlinear elliptic partial differential equations [2], Optimal control problems with constraints [3], and so on.

In this article, we first prove the deflection theorem and make it clear that if swimmers want to cross the river in the shortest time with the optimal path, the direction of the velocity u should remain the same. Then, we obtain the general solution of the total motion time T in horizontal and vertical directions by simultaneous velocity-displacement equations. Then, the constant velocity value of 1.2m/s and the swimmer's velocity value of 1.5m/s are substituted into the equation in order to get the angle θ between the swimmer's velocity and the flowing direction, the angle θ between the optimal path and the horizontal direction, and the minimum time T. Finally, through further analysis, we get that the sufficient condition for realizing this scheme. When the flow velocity distribution is a piecewise constant function and is centrosymmetric about (500,600), By combining the horizontal and vertical velocity displacement equations, we obtain the optimal model of the minimum time T and make it clear that the velocity deflection angle θ_1,θ_2 of the swimmer in the two velocity zones respectively. Then obtain the angle $\alpha,\ \beta$ between the broken line at both ends and the horizontal direction. Then we use Matlab programming to finally obtain the shortest time T. When the flow rate is a parabola symmetric about y=600m the key to solving this problem lies in the selection of the swimmer's initial speed direction θ_0. We use a computer program to approximate the initial speed direction of the swimmer θ_0 by using dichotomy and interpolation respectively. Then we obtained the shortest time T.

2. Application of Functional Analysis in Solving River Crossing Problems under Dynamic Water Flow Conditions

2.1. Angle Theorem and River Crossing Strategy

First of all, make a general assumption that the flow rate of a river v(y) is a function of the distance y from the starting point to the offshore, and specify that the swimmer's speed size u remains constant throughout the path, Let's first prove the following lemma: If the function $v(y) \equiv v_0$ The direction of speed u remains the same in the optimal path for swimmers to cross the river in the shortest time.

We first describe the problem in terms of the following equations [4]:

$$\begin{cases} \dfrac{dx}{dt} = ucos\big(\theta(y)\big) + v(y) \\ \dfrac{dy}{dt} = usin\big(\theta(y)\big) \end{cases} \tag{1}$$

Where the boundary conditions are satisfied:

$$x(0) = 0, x(T) = L, y(0) = 0, y(T) = H \tag{2}$$

Take the admissible set of functions:

$$S = \{\theta(y)|\theta \in C^{(2)}_{[0,H]}, y(0) = 0, y(T) = H\} \tag{3}$$

Demand:

$$minT = \int_0^H \frac{dy}{usin\big(\theta(y)\big)} \tag{4}$$

Make:

$$\int_0^H \frac{ucos\theta + v(y)}{usin\theta} dy = L \tag{5}$$

The solution of the above admissible function set S is essentially a functional extremum problem with constraints.

The Hamilton function corresponding to this problem is given [5]:

$$H = F + \lambda G = \frac{1}{usin(\theta(y))} + \lambda \frac{ucos(\theta(y)) + v(y)}{usin(\theta(y))} \tag{6}$$

Next, we use the Euler equation [6] to find the resident function of this function:

$$\frac{\partial H}{\partial \theta} = \frac{-cos\theta - \lambda(u + vcos\theta)}{usin^2\theta} \tag{7}$$

From the above formula we can get:

$$\forall y \in [0, H], \begin{cases} \lambda(usec(\theta(y)) + v(y)) + 1 = 0 \\ \lambda(usec\theta_0 + v_0) + 1 = 0 \end{cases} \tag{8}$$

Namely:

$$u\left(\sec\left(\theta(y)\right) - sec\theta_0 + v(y) - v_0\right) = 0 \tag{9}$$

When: $v(y) \equiv v_0$ gain: $\sec\left(\theta(y)\right) = sec\theta_0$

Sum up, we can get:

If function $v(y) \equiv v_0$, the direction of speed u remains the same in the optimal path for swimmers to cross the river in the shortest time.

We call this the declination theorem.

In this case, if the condition $v(y) \equiv v_0$ is satisfied. Then, we can keep the direction of the velocity u constant from the declination theorem, then we can get the following equation [7]:

$$\begin{cases} vT - uTcos\theta = L \\ uTsin\theta = H \end{cases} \tag{10}$$

Collation available:

$$(v^2 - u^2)T^2 - 2vLT + (L^2 + H^2) = 0 \tag{11}$$

Solving this binary first-order equation gives:

$$\mathrm{T} = \frac{\mathrm{vL} \pm \sqrt{\mathrm{v}^2\mathrm{L}^2 - (\mathrm{v}^2 - \mathrm{u}^2)(\mathrm{L}^2 + \mathrm{H}^2)}}{\mathrm{v}^2 - \mathrm{u}^2} \tag{12}$$

Plug in the data from the problem and you get: $\mathrm{T} = 800.432\mathrm{s}$

The negative solution T=-3763.3958s has been dropped.

Also due to: $\cos\theta = \frac{\mathrm{L} - \mathrm{vT}}{\mathrm{uT}} \approx 0.0328$

We gain: $\theta \approx 88.1156^{\circ}$

Let the Angle between the optimal path and the X-axis in this case be α, then according to the sine theorem:

$$\frac{u}{sin\alpha} = \frac{v}{sin(\theta - \alpha)} \tag{13}$$

The results are as follows:

$$\tan\alpha = \frac{\sin\theta}{\frac{\mathrm{v}}{\mathrm{u}} + \cos\theta} \approx 1.2000 \tag{14}$$

From this, we get: $\alpha = \arctan(1.2000) \approx 50.1944°$

According to the above results, we get the optimal path diagram as shown in figure 1:

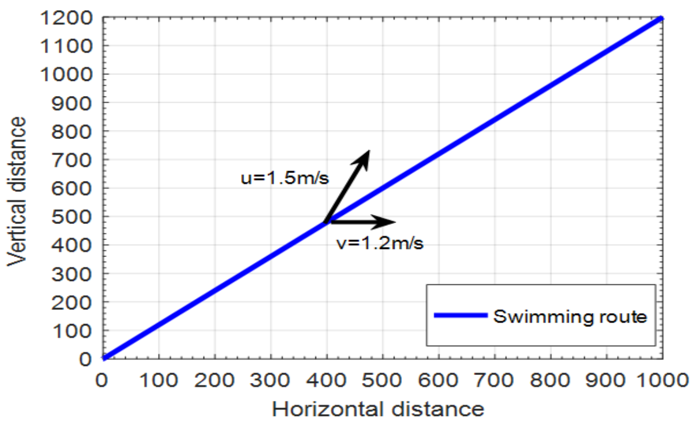

Figure 1. Optimal path diagram.

In this case, we have found that the optimal route is a straight line connecting the starting point and the ending point. The direction of this straight line is the synthesis direction of human swimming speed and water flow speed. Let's discuss the speed requirements of the contestants to reach the end along this path: From geometric knowledge, we know that human swimming speed must meet the following conditions [8]:

$$u \geq v\sin\theta$$
$$\sin\theta = \frac{H}{\sqrt{H^2 + L^2}} \quad (15)$$

By substituting H = 1200m, L = 1000m, v =1.2m/s, we get:$u_{min} = 0.92186553\, m/s$

That is when the swimmer's swimming speed meetsu $\geq u_{min}$ participants can then follow the best possible route to reach the finish line in the least possible time. The above formula is the sufficient condition that the optimal path can be implemented.

2.2. Optimizing River Crossing Strategy under Symmetric Flow Speed

According to the symmetrical distribution of river velocity, it can be seen that the corresponding path on the X-axis interval $(0.\frac{L}{2})$ and $(\frac{L}{2}, L)$ is centrosymmetric concerning the point $(\frac{L}{2}, \frac{H}{2})$, Therefore, we simplify the analysis of this problem to a part of the interval $(0,\frac{L}{2})$. As shown in figure 2:

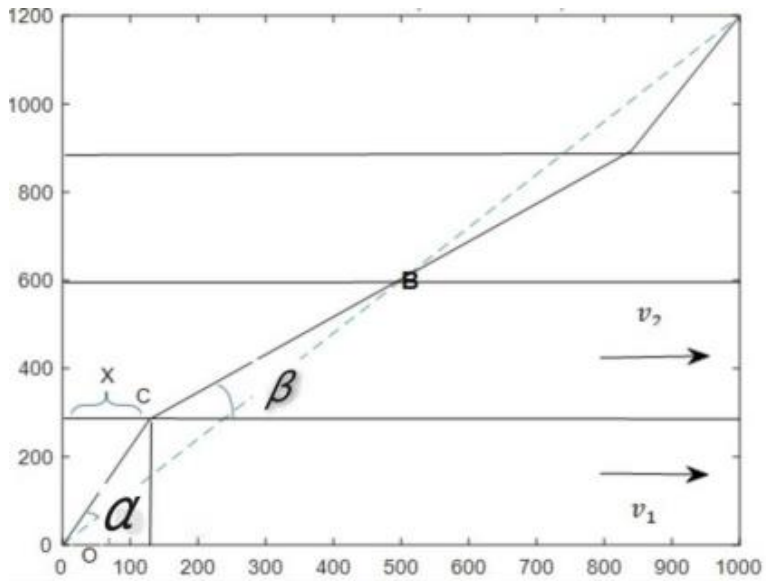

Figure 2. Simplified analysis diagram.

First, according to the Y-axis interval of$\left(\frac{0,H}{4}\right)and\left(\frac{H}{4}, \frac{H}{2}\right)$, the river velocity is fixed. By the declination theorem, It can be seen that the swimmer's velocity deflection Angle in these two intervals is fixed respectively, When the river speed isv_2, the declination Angle corresponding to the swimmer's speed u is set toθ_1, θ_2, At the same time, the time it takes to swim through these two sections is T_{11}, T_{12}, And set the horizontal coordinate of the swimmer passing through the line $y = \frac{H}{4}$ to X. We can list the following equations [9]:

$$\begin{cases} u\sin\theta_1 T_{11} = H_1 \\ \underline{v_1 T_{11} + uT_{11}\cos\theta_1 = X} \\ v_2 T_{12} + uT_{12}\cos\theta_2 = \frac{L}{2} - X \\ u\sin\theta_2 T_{12} = \frac{H}{2} - H_1 \end{cases} \quad (16)$$

The equation is reduced to:

$$\begin{cases} (v_1^2-u^2)T_{11}^2-2v_1XT_{11}+(X^2+H_1^2)=0 \\ (v_2^2-u^2)T_{12}^2-2v_2(\frac{L}{2}-X)T_{12}+[(\frac{L}{2}-X)^2+(\frac{H}{2}-H_1)^2]=0 \end{cases} \tag{17}$$

Where $H_1=\frac{H}{4}$ and X is required to obtain the minimum value of $T=2(T_{11}+T_{12})$. According to the equation (17), the Matlab program can be finally solved:

$$\begin{gathered} T_{11}=203.3704s \\ T_{12}=122.5416s \\ T_{min}=811.8240s \\ X=122.6412m \end{gathered} \tag{18}$$

By substituting corresponding results into equation (17), we can get:

$$\begin{cases} cos\theta_1=\dfrac{X-v_1T_{11}}{uT_{11}}=-0.17 \\ cos\theta_2=\dfrac{\frac{L}{2}-X-v_2T_{12}}{uT_{12}}=-0.15 \end{cases} \tag{19}$$

It can be obtained that:

$$\begin{cases} \theta_1=100.3523° \\ \theta_2=99.0850° \end{cases} \tag{20}$$

The following diagram shown in figure 3 is our optimal path graph based on the above parameters:

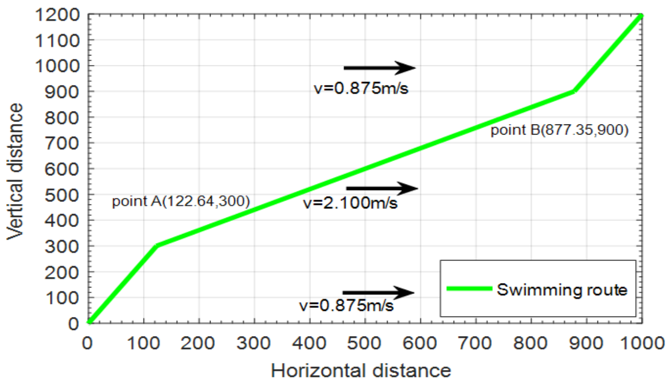

Figure 3. Optimal path diagram.

Let's say the Angle between the two broken lines and the X-axis is α,β respectively, then the sine theorem can be obtained:

$$\begin{cases} \dfrac{u}{\sin\alpha}=\dfrac{v_1}{\sin(\theta_1-\alpha)} \\ \dfrac{u}{\sin\beta}=\dfrac{v_2}{\sin(\theta_2-\beta)} \end{cases} \tag{21}$$

Sorting out this equation set we get:

$$\begin{cases} \tan\alpha=\dfrac{\sin\theta_1}{\frac{v_1}{u}+\cos\theta_1}\approx 2.4371 \\ \tan\beta=\dfrac{\sin\theta_2}{\frac{v_2}{u}+\cos\theta_2}\approx 0.7950 \end{cases} \tag{22}$$

According to this equation set, we can get:

$$\begin{cases} \alpha=\arctan(2.4371)\approx 67.6905° \\ \beta=\arctan(0.7950)\approx 38.4847° \end{cases} \tag{23}$$

We have obtained that the optimal route is a broken line, and we will discuss this line in sections: First, when the contestant enters the first-class speed zone (water velocity is 0.875m/s), the speed of the contestant must meet the following conditions according to geometric knowledge [10]:

$$\begin{cases} u_1 \geq v_1 \sin\theta_1 \\ \sin\theta_1 = \dfrac{H_1}{\sqrt{H_1^2 + X^2}} \end{cases} \tag{24}$$

Substituting $v_1 = 0.975\text{m/s}, H = 300m \ X = 122.6412m$ the solution is: $u_{1(min)} = 0.809934\, m/s$

Secondly, when the participant enters the second velocity zone (the water velocity is 2.100m/s), which can be obtained by geometric knowledge, the participant's speed must meet the following conditions [10]:

$$\begin{cases} u_2 \geq v_2 \sin\theta_2 \\ \sin\theta_2 = \dfrac{\frac{H}{2} - H_1}{\sqrt{(\frac{L}{2} - X)^2 + (\frac{H}{2} - H_1)^2}} \end{cases} \tag{25}$$

Substitute into $H_1 = 300m, L = 1000m, X = 122.6412m$, $v_2 = 2.12.100\, m/s, H = 1200m,$ the solution is: $u_{2(min)} = 1.3068411\, m/s$

In summary, when the participant's speed meets the following conditions:

$$\begin{cases} u \geq 0.809934, t \in (0,203.3704) \cup (608.4536,811.8240) \\ u \geq 1.3068411, t \in (203.3704,608.4536) \end{cases} \tag{26}$$

This model can select the appropriate optimal path to minimize the time of the race. The above formula is the sufficient condition that the optimal path can be implemented.

We carried out a sensitivity analysis by controlling other variables unchanged, respectively, the speed of small floating runners u(m/s), river length L(m), and width H(m), and obtained the following data shown in TABLE I, TABLE II and TABLE III:

Table I. Influence of changing velocity u on time t when other parameters are constant.

u(m/s)	1.2	1.3	1.4
T(s)	1152.9	1108.4	886.462
u(m/s)	1.5	1.6	1.7
T(s)	811.824	753.81	706.72

Table II. Influence of changing the size of H on time T when other parameters are constant.

H(m)	1020	1080	1140
T(s)	704.5598	736.5914	777.5914
H(m)	1200	1260	1320
T(s)	811.8240	846.9231	887.3229

Table III. Influence of changing the size of L on time T when other parameters are constant.

L(m)	Time(s)
900	830.1997
950	819.7473
990	813.2207
1000	811.8240
1010	810.5167
1030	808.1616
1040	807.1095
1050	810.3624
1060	813.2558
1100	818.8349

2.3. Analysis of Optimal River Crossing Path under Non-linear Flow Speed

The flow rate of a river is a quadratic function of distance from shore, which means that the direction of a swimmer's speed will change in any y interval. Since the change of speed direction has continuity, the key to the distribution function $\theta(y)$ that affects the speed direction of the entire swimmer is the selection of the initial speed directionθ_0. In equation (9), by substituting$v_0 = 0$, we can gain:

$$\sec(\theta(y)) = \sec\theta_0 - \frac{v(y)}{u} \tag{27}$$

Our main task is to find the value of the parameterθ_0. The condition for constraint θ_0 is: The swimmer should be allowed to reach L in the horizontal direction after completing the process of vertical path H. That is, from formula (5), it can be obtained:

$$\int_0^H \frac{\frac{v(y)}{u}\ sec\theta + 1}{tan\theta}\ dy = L \tag{28}$$

From the trigonometric function formula:

$$tan^2\theta = sec^2\theta - 1 \tag{29}$$

We can get:

$$\int_0^H \frac{\frac{v(y)}{u}\ [\frac{1}{cos\theta_0} - \frac{v(y)}{u}] + 1}{\sqrt{[\frac{1}{cos\theta_0} - \frac{v(y)}{u}]^2 - 1}}\ dy = L \tag{30}$$

Among:

$$v(y)=\frac{2.1}{360000}\ y(1200 - y), 0 \le y \le 1200 \tag{31}$$

Equation (30) is the constraint equation ofθ_0. With this integral equation, we can solve for the value of $\cos\theta_0$ However, because this integral equation is too complex to give a specific analytical solution, we choose to approximate the value of zero that meets the reasonable error range through a computer program. Here are the two ways we will give the value of $\cos\theta_0$Method 1: A Bisection Algorithm is used to select the appropriate $\cos\theta_0$his solution of the integral equation of equation (30) is approached continuously until a value of $\cos\theta_0$s obtained that satisfies the accuracy.

We consider 10^{-6} as the size of the differential interval and give an integral function that satisfies the iterative difference less than10-6(that is, the difference between this integral and the last integral result in the cycle). The integral function is then assigned possible $\cos\theta_0$rameters by a dichotomy. Finally, by continuously approximating the target value target=L, we can obtain a $\cos\theta_0$hat satisfies the accuracy.

At the end of the program, we substitute the final $\cos\theta_0$lue into the integral on the left of the equation (30) and verify that the floating range of the integral result and L=1000m is less than10^{-6}, which can be considered to be a very accurate result.

Through this operation, we get the value that satisfies the range of differential interval less than10^{-6}, and integral error less than 10^{-6} is:$\cos\theta_0 = 0.115994511$

And thus can be obtained:$\theta_0 = \arccos(0.115994511) = 83.3390°$

Method 2: In this method, our idea is to treat the flow rate of the river in each small interval as a constant through the interpolation method, then we can know that the velocity direction in the interval is a fixed value by the declination theorem. We divide the width of the river surface [0, H] into n equals, interpolate each small interval segment, and record$0 = y_0 < y_1 < \cdots < y_n = H$, Step:$d = \frac{H}{n}$

We assume that the speed of the inner swimmer is a constant value for each small segment $[y_{i-1}, y_i]$ and take the mean value of the water flow in $[y_{i-1}, y_i]$ to approximatev_i, that is:

$$v_i = \frac{1}{2} \times [\frac{2.1}{360000} \times y_{i-1} \times (1200 - y_{i-1}) + \frac{2.1}{360000} \times y_i \times (1200 - y_i)] \tag{32}$$

Let the Angle of the swimmer swimming in each small interval beθ_i, and require the initial θ_0 parameter value to make the river crossing time T shortest, the following optimization models [11] are listed: $\min T = \sum_{i=1}^{\frac{H}{d}} t_i$

$$\text{s.t.} \begin{cases} u t_i \sin\theta_i = d \\ \sum_{i=1}^{\frac{H}{d}} t_i\,(u\cos\theta_i + v_i) = 1000 \\ \theta_i \in [0, \pi] \end{cases} \tag{33}$$

Let's take d=10^{-3}, and that gives us our final answer:

$$\cos\theta_0 = 0.115994511$$
$$\theta_0 = \arccos(0.115994511) = 83.3390° \tag{34}$$

To sum up, it can be obtained according to equation (14):

$$\theta(y) = \operatorname{arcsec}[\frac{1}{0.115994511} - \frac{2.1y(1200-y)}{720000}] \tag{35}$$

Substitute equation (30) into equation (4) to get

$$T_{min} = \int_0^{1200} \frac{dy}{2\sqrt{1 - \frac{1}{[\frac{1}{\cos\theta_0} - \frac{2.1y(1200-y)}{720000}]^2}}} \tag{36}$$

This definite integral can be computed: $T_{min} \approx 604.86s$

Next, we use the phase trajectories of the two-dimensional differential equations to give the specific optimal path diagram.

First, by substituting our desired$\theta(y)$function expression (35) into equation (1), we get:

$$\begin{cases} \frac{dx}{dt} = \frac{2}{\frac{1}{\cos\theta_0} - \frac{2.1y(1200-y)}{360000}} + \frac{2.1y(1200-y)}{360000} \\ \frac{dy}{dt} = 2\sqrt{1 - \frac{1}{(\frac{1}{\cos\theta_0} - \frac{2.1y(1200-y)}{720000})^2}} \end{cases} \tag{37}$$

Input this two-dimensional differential equation set into Matlab for processing and the following function image of y=y(x)can be obtained, that is, the optimal path diagram which is shown in figure 4 for crossing the river:

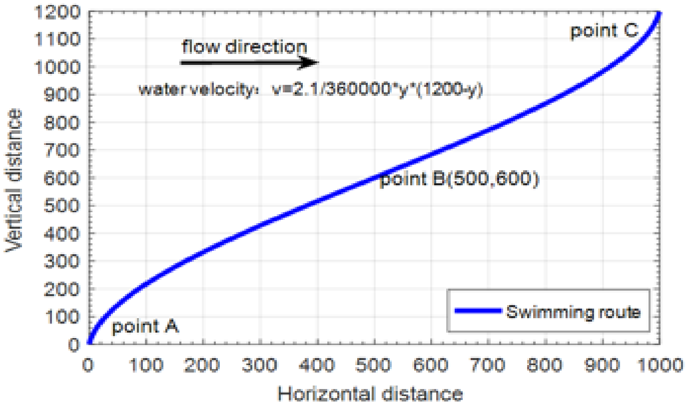

Figure 4. Optimal path diagram.

In this case, for the swimmer to follow the optimal path at any time t, the speed must be satisfied:

$$u(y) \geq v(y)\sin(\theta_1(y)) \tag{38}$$

By differential equations [12]:

$$\begin{cases} \dfrac{dx}{dt} = u(y)cos(\theta(y)) + v(y) \\ \dfrac{dy}{dt} = u(y)sin(\theta(y)) \end{cases} \tag{39}$$

Gain:

$$\tan(\theta_1(y)) = \frac{dy}{dx} \tag{40}$$

From this, we derive:

$$\sin(\theta_1(y) = \frac{dy}{\sqrt{(dx)^2 + (dy)^2}} = \frac{\frac{dy}{dt}}{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}} = \frac{u(y)sin(\theta(y))}{\sqrt{[u(y)cos(\theta(y)) + v(y)]^2 + [u(y)sin(\theta(y))]^2}} \tag{41}$$

By substituting this formula into equation (38), we get:

$$u(y) \geq v(y)\frac{u(y)sin(\theta(y))}{\sqrt{[u(y)cos(\theta(y)) + v(y)]^2 + [u(y)sin(\theta(y))]^2}} \tag{42}$$

From equation (27), we get:

$$u(y) \geq v(y)\frac{u(y)sin(\theta(y))}{\sqrt{[u(y)cos(\theta(y)) + v(y)]^2 + [u(y)sin(\theta(y))]^2}} \tag{43}$$

Substitute into initial condition $v_0 = 0$, we gain:

$$\sec(\theta(y)) - \sec(\theta_0) = \frac{-v(y)}{u(y)} \tag{44}$$

Namely:

$$u(y) = \frac{v(y)}{\sec(\theta_0) - \sec(\theta(y))} \tag{45}$$

Sort out:

$$u(y) \geq v(y)\frac{\frac{\sin(\theta(y))}{\sec(\theta_0) - \sec(\theta(y))}}{\sqrt{\frac{(\cos(\theta(y)) + \sec(\theta_0) - \sec(\theta(y)))^2 + (\sin(\theta(y)))^2}{(\sec(\theta_0) - \sec(\theta(y)))^2}}} \tag{46}$$

Finally, we get that u(y) should satisfy the following inequality:

$$u(y) \geq \frac{v(y)\sin(\theta(y))}{\sqrt{(\cos(\theta(y)) + \sec(\theta_0) - \sec(\theta(y)))^2 + (\sin(\theta(y)))^2}} \tag{47}$$

When this condition is always true, the model can select the appropriate optimal path to minimize the race time. The above formula is the sufficient condition that the optimal path can be implemented.

We also carried out a sensitivity analysis by controlling other variables unchanged, respectively, the speed of small floating runners, river length, and width. After obtaining multiple sets of data, the cubic spline interpolation method was adopted based on Matlab to carry out data fitting, and the following change curve was obtained, as shown in figure 5.

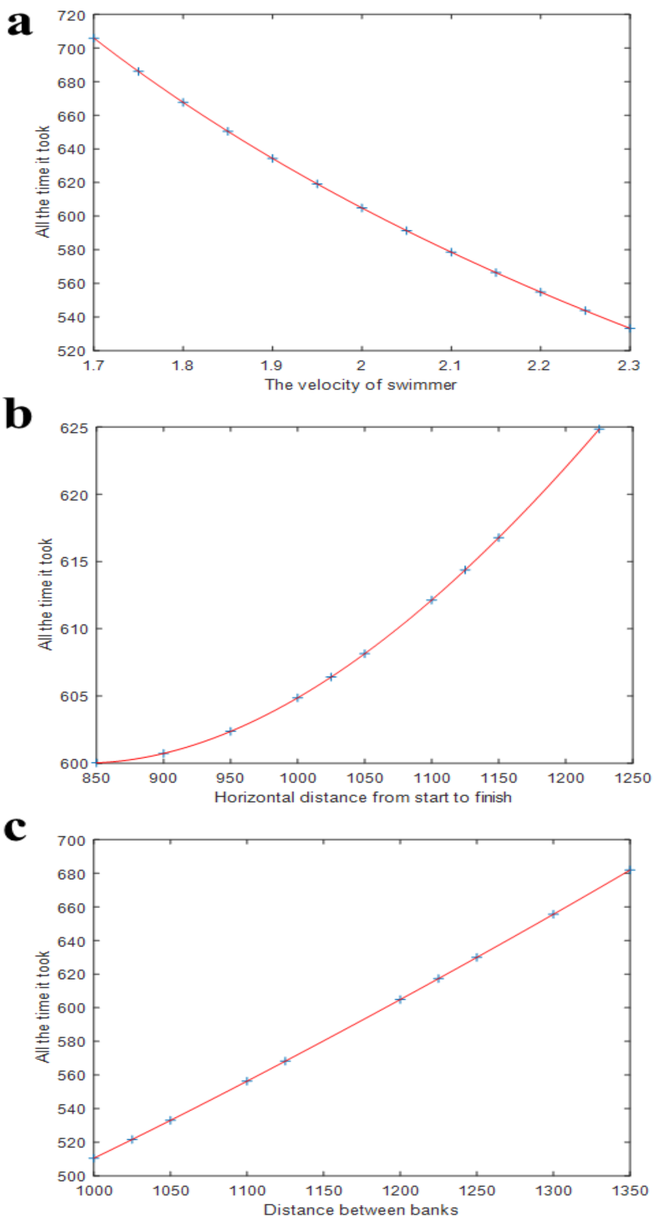

Figure 5. Sensitivity analysis graph. a speed of small floating runners b river length c width

3. Conclusions

This study reveals a method called the Hamilton-Lagrange multiplier which is based on the variational method and functional analysis theory to solve the constrained functional extremum problem by taking the swimmer's velocity direction θ(y) and the distribution of water velocity v(y) as the functional parameters, and taking the swimmer's total horizontal displacement as the constraint condition. By solving the Euler equation, a universal continuous-time optimal control model is established. Using dichotomy and numerical analysis, the optimal trajectory and direction control law of swimming speed is solved, and the optimal path scheme to reach the endpoint is proposed.

We first prove the deflection theorem and make it clear that if swimmers want to cross the river in the shortest time with the optimal path, the direction of the velocity u should remain the same when flow velocity distribution is a constant function.

Then, we obtain the general solution of the total motion time T in horizontal and vertical directions by simultaneous velocity-displacement equations. Then, the constant velocity value of 1.2m/s and the swimmer's velocity value of 1.5m/s in problem 1 are substituted into the equation, and the Angle between the swimmer's velocity is $\theta = 88.11^{\circ}$, The Angle between the optimal path and the horizontal direction is $\theta = 50.19^\circ$, and the minimum time is T=800.43s. Finally, we get that the sufficient condition for realizing this scheme is that the speed of swimmers cannot be lower than 0.9219m/s.

When the flow velocity distribution is a piecewise constant function and is centrosymmetric about (500,600), to simplify the problem, we discuss it at$(0,\frac{L}{2})$, It is known by the declination theorem, in the interval$(0,\frac{H}{4})$ and$(\frac{H}{4},\frac{H}{2})$, the velocity Angle of the person is two fixed values respectively. We obtain the optimal model of the minimum time T and make it clear that the velocity deflection Angle of the swimmer in the two velocity zones is $\theta_1 = 100.35^\circ$ and $\theta_2 = 99.08^\circ$ respectively, The Angle between the broken line at both ends and the horizontal direction is $\alpha = 67.69^\circ$, $\beta = 38.48^\circ$ Then we use Matlab programming to finally obtain the shortest time T=811.82s, draw the human swimming path diagram, analyze the sufficient conditions under the premise of the optimal path.

When the flow rate is a parabola symmetric about y=600m, there is continuity due to the change of velocity direction, Therefore, the key to solving this problem lies in the selection of the swimmer's initial speed directionθ_0. We use a computer program to approximate the initial speed direction of the swimmer $\theta_0 = 83.34^\circ$ by using dichotomy and interpolation respectively. Besides, the Angle of the swimmer's speed direction at each offshore distance y is:

$$\theta(y) = \text{arcsec}\left(\frac{1}{0.115994511} - \frac{2.1y(1200-y)}{720000}\right) \tag{48}$$

By substituting it into the functional equation originally established, the shortest time T=604.8619s can be obtained by integration. By using the method of drawing the phase track of a two-dimensional equation, we simulate the optimal path image of the swimmer and give sufficient conditions for realizing this path.

Finally, this paper carried out the sensitivity analysis and error analysis of the model, and unexpectedly found that in the second problem, when the horizontal distance of swimming is near L=1040m, the lowest point of the total time will appear.

The feature of this paper is that the optimal control model is established based on the functional analysis method, which can flexibly deal with various distributed parameters, and can be well qualified for the task of describing continuous problems with high accuracy and giving optimal control functions. In addition, the model can also be extended to other fields such as transportation, finance, and energy, and has high application value.

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