

Analysis of Principles and Applications of Chaos Theory

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Abstract. As early as Aristotle's period, people were trying to find rules to define nature. Centuries later, there came Newton's Laws of Motion, it was 'omnipotent' in explaining everything under gravitation, up to early twentieth century. Then, Einstein proposed theory of special relativity (responsible for everything else without gravitation), and general relativity (generalizing gravitation and special relativity). Nevertheless, along with order there is chaos. Chaos theory was not explicitly realized until late in the twentieth centuries, and that was when people realized that chaos is also capable of explaining the world. Computers are good with handling large data calculations, which the development of chaos theory benefits from. This study discusses some fundamental ideas of chaos theory, and focus on how they can be used to predict near futures in complex systems such as weather forecasting, stock markets, and transport systems. According to the analysis, there are also limitations, which again relates to computing technologies. However, the future is bright as advanced technologies await in the big data era.

Keywords: Chaos; dynamics; complex systems; chaotic systems.

1. Introduction

A small experiment on the calculator may be an easy entry to chaos theory [1, 2]. Consider an easy function, for example, a recurrence relation $2x^2 - 1$ and typing it in a calculator, with any initial value, example, 0.89, and the values, keep pressing the 'equals' button. For however many iterations, the result still seems be random (seen from left upper panel of Fig. 1). If one still uses the same function, changes the initial value to, for example 0.892, then the results are different (seen from right upper panel of Fig. 1). Different from the chaos observed for the above function, consider another recurrence relation $x^2 - 1$. Still setting the initial value to be 0.89 (left lower panel of Fig. 1) and 0.892 (right lower panel of Fig. 1) respectively, although at first the number seems to be 'jumping around' randomly, but after a few iterations, it will oscillate between values -1 and 0. From this small simulation an important characteristic is shown sensitivity to initial conditions.

Another classic way to easily 'see' chaos is via a double pendulum [3]. When the angle of displacement $\theta \ll 1$, the motion of a double pendulum can be seen as the same as a simple harmonic oscillator, shown by the linear relationship between the two angles. However at large angles, its motion becomes chaotic and unpredictable.

The stability of solar system was a fundamental physics question proposed by King Oscar II of Sweden in 1887. Henri Poincaré was one of who attempted the problem (and won the prize though he did not answer it directly) [2]. The most significant achievement through this was the invention of topology. As Poincaré was looking at an idealized three-body-motion problem (Hill's reduced model), the trajectory he depicted were now known as 'chaotic' [2]. From Poincaré's studies fundamental ideas such as phase space and Poincaré's sections were introduced [2]. Edward Lorenz of the Massachusetts Institute of Technology (MIT) is famously known as the founder of modern chaos theory [2, 4]. In 1963, he published the paper with details of behaviors found from computer simulation this was the birth of chaos theory. By looking at convections, Lorenz proposed his famous equation - Lorenz equations (stated explicitly later), and by trajectory of his equations, computer could plot the Lorenz attractor [2, 4]. By accident, he discovered what is called the 'butterfly effect' [4]. David Ruelle, together with Floris Takens, took the idea of attractors further and proposed 'strange attractors', a term Ruelle coined in 1971 [2, 4]. Between late 1950s and early 1970s, 'fractal' (a name given in 1975), a new branch in mathematics, was evolved by Benoit [2]. It was in the 70s when chaos and fractal appeared to be related, where fractals led a new way to geometry of chaos.

The motivation of this paper is see how chaos theory, a highly ‘pure’ mathematical idea, is in fact utilized in monitoring real-life related scenarios – weather forecasting, stock market monitoring, and in transportation systems. Concepts introduced in section two will support a better understanding to the aspects of chaos applicable to the real life scenarios, as well as the charectistics that make its applications possible. Last but not least, imitiations in the chaos theory are briefly stated.

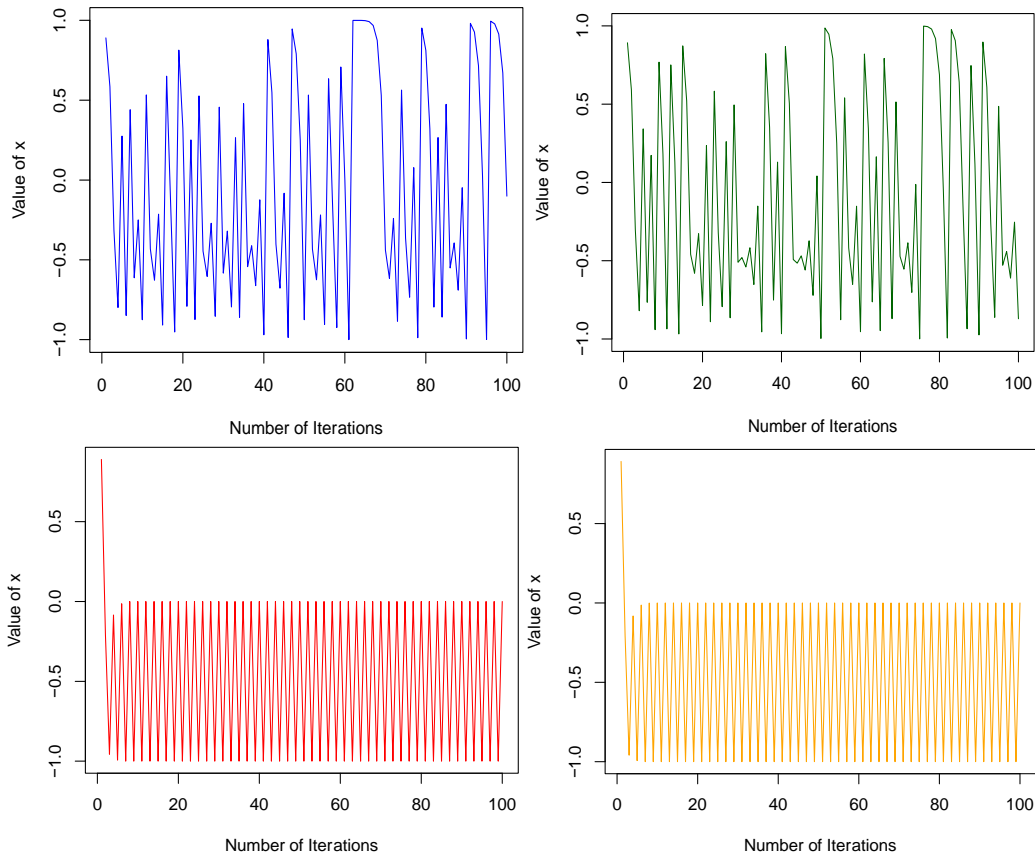


Fig. 1 100 Iterations of function $f(x)=2x^2-1$ with initial value 0.89 (upper left) and 0.892 (upper right); $f(x)=x^2-1$ with initial value 0.89 (lower left) and 0.892 (lower right).

2. Basics of Chaos Thoery

2.1. Definition of Chaos

Chaos usually refers to stochastic behavior of a system, and associates with unpredictable activities. There are many quantitative mathematical definitions of ‘chaos’, or of ‘chaotic’ system. Below are three commonly used definitions of chaos, of continuous maps. Let map $f: X \rightarrow X$ on some state space X be a continuous map, then this map is chaotic if...

Definition 1 (Devaney’s definition): in an invariant subset $Y \subseteq X$ such that:

1. f is topologically transitive in Y ;
2. The set of points that initiate period orbits densely in Y ;
3. Dependence on initial conditions is sensitive and weak [5, 6].

Definition 2 (Block and Coppel’s definition): $\exists m \in \mathbb{N}$ and $Y \subseteq X$ such that Y is a compact f^m -invariant subset, and there exists a continuous surject $g: Y \rightarrow \Sigma$ where $g \circ f^m = \sigma \circ g$ on Y [6].

Definition 3 (Li and Yorke’s definition): in an uncountable subset $S \subseteq X$ that satisfies that:

$\forall p, q \in S$ with $p \neq q$, [6, 7]:

1. $\limsup_{n \rightarrow \infty} |f^n(p) - f^n(q)| > 0$;
2. $\liminf_{n \rightarrow \infty} |f^n(p) - f^n(q)| = 0$; $\forall p \in S$ and periodic point $q \in X$,
3. $\limsup_{n \rightarrow \infty} |f^n(p) - f^n(q)| > 0$.

The above definitions are of one-dimensional chaos of interval maps (and maps are considered smooth for convenience) [8]. It may be helpful to consider some examples of functions that generate chaos and ones that do not. For example, a sine map $x_{i+1} = S(Xi) = p \sin(\pi x_i)$, with control parameter $p \in [0,1]$ produces chaos. This map originates from the sine function, which maps input angles in interval $[0,1]$ into $[0,1]$ [9]. On the other hand, simple linear functions such as $f(x) = 2x$ is not chaotic as it has a clear, predicatable pattern. Interval maps that decrease monotonically are also not chaotic. This is because they converge to a unique equilibrium point, hence is a predictable and orderly behavior [9].

2.2. Lorenz Equations

A famous example of chaos has always been the ‘Butterfly Effect’ (seen from Fig. 2). This butterfly-like graph was a result of the three-dimensional system called the Lorenz equations

$$\begin{cases} \frac{dx}{dt} = \sigma(-x + y) \\ \frac{dy}{dt} = -xz + rx - y \\ \frac{dz}{dt} = xy - bz \end{cases} \quad (1)$$

where for convenience Lorenz used $\sigma = 10$, $r = 28$, and $b = 8/3$. But in general, these constants can be any number [10-12]. Simple properties of the Lorenz equations are [13, 14]:

Nonlinearity – xy and xz are nonlinear;

Symmetry – equations are invariant under the transformation $(x, y) \rightarrow (-x, -y)$;

Fixed Points – steady state occurs when $\dot{x} = \dot{y} = \dot{z} = 0$. There are two solutions: $(x^*, y^*, z^*) = (0,0,0)$, and $(x^*, y^*, z^*) = (\pm\sqrt{r-1}, \pm\sqrt{r-1}, r-1)$

Volume Contraction – in phase space volumes contract under flow (i.e. the system is dissipative).

By the Lie derivative, volume contracts at a rate: $\frac{1}{V} \frac{dV}{dt} = \sum_i \frac{\partial \phi_i}{\partial \phi_i}, i = 1,2,3, \phi_1 = x, \phi_2 = y, \phi_3 = z$. From calculations yields $V(t) = V(0)e^{-(P+1+b)t}, P > 0, b > 0$.

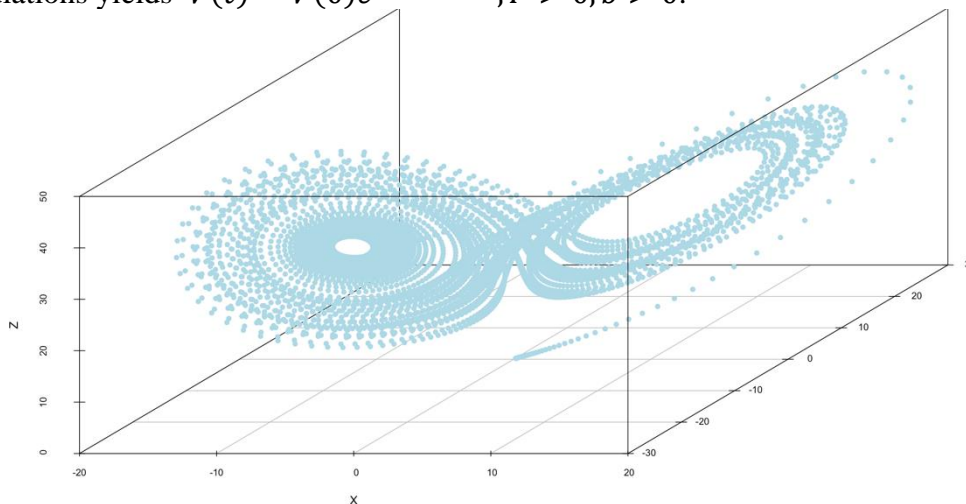


Fig. 2 A Lorenz attractor shown in 3D with initial values: $\sigma = 10$, $r = 28$, and $b = 8/3$.

2.3. Strange Attractors

Lorenz attractor is an example of strange attractors. Let that at time $(t+1)$, a system is specified by the parameters:

$$\begin{cases} x_1(t+1) = F_1(x_1(t), x_2(t), \dots, x_m(t)) \\ \dots \\ x_m(t+1) = F_m(x_1(t), x_2(t), \dots, x_m(t)) \end{cases} \quad (2)$$

Assuming F_1, F_2, \dots, F_m are continuously differentiable, and assign initial values $x_1(0), x_2(0), \dots, x_m(0)$ to each of the parameters [13-15]. Let F be a map in m -dimensions such that $F: x_1, \dots, x_m \mapsto F_1(x_1, \dots, x_m), \dots, F_m(x_1, \dots, x_m)$. In m -dimensions, a bounded set A is a strange attractor for map F if there exists m -dimensional set U such that U is in neighborhood of A . In positive time t , point $X_t := (x_1(t), \dots, x_m(t)) \in U$ stays close to A for t large enough – A is attracting; A is a strange attractor as it is sensitively dependent on initial conditions [15]. Observing the geometric structure of the strange attractor – the trajectory seems to be intersecting itself repeatedly, but actually in three-dimensions, there is no crossings occurring [11]. Lorenz has inferred, although the Lorenz attractor seems to be a single surface, it is actually “an infinite complex of surfaces”. This is what is called a fractal. A strange attractor is defined to be sensitively dependent on initial conditions [13].

2.4. Fractals

Fractals is defined as a set of points from iterations of nonlinear systems. The input value depends on the previous output value [16]. For example in the Lorenz attractor, the precision of trajectory of a point depends on the precision of the trajectory of the previous point. The two important properties of fractal are: self-similarity and fractional dimensions. Fractal dimensions refer to dimensions that fall between 1, 2, and 3 dimensions [10]. This is a characteristic that many real-life systems hold, hence why chaos theory is applicable in these areas. The typical results are shown in Fig. 3 [17].

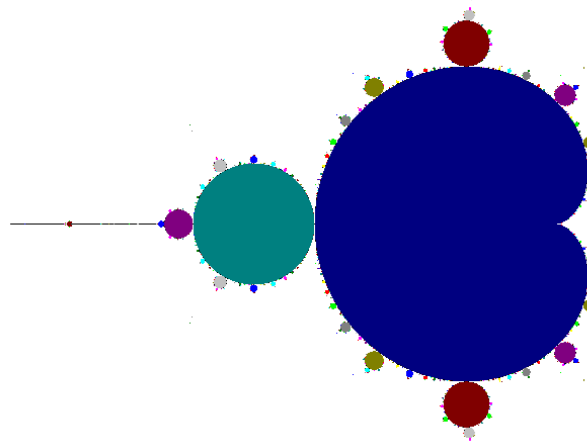


Fig. 3 The Mandelbrot Set “gingerbread man” [17]. Equation of the plot is $z_{n+1} = z_n^2 + c$ starting with $z=0$.

3. Applications

In this section, applications in three major fields are discussed - namely in weather forecasting, stock markets, and transportation systems. All three are considered complex systems with nonlinear dynamics. A major characteristic that all contains is the sensitivity to initial conditions. The rapid development of computers plays an unignorable role for allowing chaos theory to become application in these real-life scenarios. So far, the theory is mostly used to predict future events, or at least to allow people to foresee the chaos.

3.1. Weather Forecasting

As a meteorologist, Edward Lorenz discovered chaos (theory) when rounding decimal points of variables on the computer and discovering completely different results [2]. Chaos theory is widely applicable in weather forecasting. Weather prediction is essential for everyday-life activities, and contributes greatly in industries like agriculture, transportation, energy, etc. Traditionally, numerical weather prediction (NWP) is used. The atmospheric equations of motion can be derived from fluid dynamics and first law of thermodynamics [18]. It is as follows:

$$\frac{Du}{Dt} - \frac{uv \tan \phi}{r} + \frac{uw}{r} = -\frac{1}{\rho r \cos \phi} \frac{\partial p}{\partial \lambda} + 2\Omega v \sin \phi - 2\Omega w \cos \phi + v \nabla^2 u \quad (3)$$

$$\frac{Dv}{Dt} + \frac{u^2 \tan \phi}{r} + \frac{uw}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial \phi} - 2\Omega u \sin \phi + v \nabla^2 u \quad (4)$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} - g + 2\Omega u \sin \phi + v \nabla^2 w \quad (5)$$

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \quad (6)$$

$$c_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = J \quad (7)$$

$$p = \rho R_d T \quad (8)$$

$$\frac{Dq_i}{Dt} = S_i \quad (9)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is the material derivative [19].

The coupled nonlinear partial differential equations (PDE) cannot be solved analytically. Instead, by using discretized time, and taking the variables such as temperature, pressure, wind vector, etc. at all points in a 3D grid, the values of these variables can be predicted at $t+\Delta t$ [2]. It is to note that the complexity of different NWP models is based on usage for climate or weather predications, as well as area of required coverage [18]. Hence by repeating the calculation and after multiple iterations, weather in future times can be predicted. Computers are good at doing repetitive calculations, however, even for the most precise devices rounding errors still arise. Yet weather is sensitively dependent on initial conditions – any small difference (even in decimal places) will lead to complete different results, resulting in increasing inaccuracy for weather forecasts further in future days. On the other hand, a common modern technique is to repeatedly forecast weather over the same period under different initial conditions, revealing an ensemble of weather forecasts - this is helpful in revealing which conditions are more relevant [18]. Scale analysis is helpful to eliminate terms in the atmospheric equation and to identify dominant phenomena that impacts the large-scale dynamics [19].

Weather forecasting techniques, as expressed by Jaseena and Kovoov, should advance with the times. Among the different forecasting models they reviewed, hybrid models (combining statistical and artificial intelligence models), showed higher accuracy in predictions of various weather factors. Within various artificial intelligence models, ANN (artificial neural networks) and SVM (support vector machine) models are more reliable. With the large dataset from the Internet, Cloud, and other online resources, unorganized data can be collected, process and eventually transform into a forecasting model [20].

Miyoshi and Sun took a different path - instead of advancing weather forecasts, the researchers applied Lorenz’s attractor in chaos theory and by computer simulation showed that weather could be modified by small adjustments in variables [21]. Method of Control Simulation Experiment (CSE) is used, and the Observing Systems Simulation Experiment (OSSE) is extended to evaluate the impact of small perturbations on weather dynamics. The goal to achieve is not to cause irreversible changes to the environment, but rather in favor of improving problems such as climate change [21].

3.2. Stock Market

It is fair to say that finance and economics are human inventions, but it would be wrong to say people can control the happenings in it. Markets are complex and dynamic, and in many senses both systematic and random [22]. Hence financial markets can be viewed as a complex and non-linear system, where a significant characteristic is that the future is unpredictable [23].

Financial chaos occurs when unexpected event occurs in the deterministic system, i.e. the market. Although from the term ‘chaos’ it gives the impression of an unappealing event, it can also be beneficial events such as innovations of financial products, payment methods etc. [22]. Being defined as ‘deterministic system’, chaos theory is one of the tools aimed at analyzing the financial market’s

behavior, through the advancement of computational technologies and the entry to the Big Data era [24]. More advanced adaptive complex model using complex theory is composed of chaos theory alongside with complex adaptive system theory and dissipative structure theory [22].

Just like weather (Sec. 3.1), markets are sensitive to initial conditions where small changes can have profound effects on market dynamics and outcomes. One noticeable consequence of changes in initial conditions is the trigger in feedback loops. For instance, a minor shift in investment leading up to series of buys and sells, then influencing prices. All prices are coordinated by supply and demand, which further relates to interest rates. This relationship and the conclusion that growth of supply results in favor of low interest rates follow from the theory of chaos and market theory [22]. In general, market cycle has four stages in in market cycle: accumulation → mark-up → distribution phase → downtrend. Within the cycle there may be extreme points such as crisis and boom. These do not fall in the categories of order nor disorder – hence they are called ‘financial chaos’ [22].

Future market behaviors are unpredictable. However, chaos theory is still useful in the sense that it allows people to expect the unexpected, and to identify potential patterns of chaos. Investors can utilize such information to process decision-making and effectively manage risk [25]. It is essential to be able to analyze complexity of markets. One way of accurately estimating complexity within a dynamic system is by the C_0 algorithm [25]. Note that by using this algorithm, one treats the market as a fractional-order financial system. Fractional-order systems involve derivatives of non-integer orders. It provides a framework for modelling complex systems with non-integer dynamics. More specifically, it explores the notable features of memory and chaos within financial systems and captures the dynamic behaviors observed in financial systems [26]. There are different definitions of fractional derivative and integral, and the one used for the C_0 algorithm mentioned above is the Caputo method. The Caputo derivative used is defined as follows. For $\alpha \in [n - 1, n)$, the α -th derivative of $f(t)$ is [25]:

$$D^C f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^\infty \frac{(d^{(n)}f(x)/dx^{(n)})}{((t-x)^{(\alpha-n+1)})} dx \quad (10)$$

The time-delayed fractional-order financial system used is:

$$\begin{cases} D^{q_1} x_1(t) = x_3(t) + (x_2(t - \tau)) - a)x_1(t) \\ D^{q_2} x_2(t) = 1 - bx_2(t) - x_1^2(t - \tau) \\ D^{q_3} x_3(t) = -x_1(t - \tau) - cx_3(t) \end{cases} \quad (11)$$

where x_1 indicates interest rate, x_2 the demands of investments, and x_3 the index for price; order of derivatives $q_i, i \in 1,2,3, q_i \in (0,1]$ is the price index; τ defines delay in time; parameters a defines quantity of savings, b is cost per investment, and c is elasticity in market demand [25]. There is necessity for policies or control considerations due to the resulting high complexity in chaotic systems like the financial market. Parameters, derivative orders, and time delays vary for different systems. It is possible for decision-makers to alter the parameters for lower complexity. Furthermore, the researchers hold the opinion that introducing delays in investment demand and interest rate may induce chaos within the system [22, 27].

As contradicting it may be (since markets are considered unpredictable), the idea that market prices exhibit fractal properties over time allows fractal analysis to be used for market forecasting. In the book ‘The Fractal Geometry of Nature’ by Mandelbrot, a hypothesis states that self-affinity is displayed financial time series. Self-affinity encompasses the same idea of self-similarity but allows more flexibility in degrees of similarity across scales [28]. Self-similarity is a key to the state of present financial market – the process of self-replication is chaotic. It also gives the answer to whether stock exchanges expand or contract, and how markets manage capital flows and direct shares.

In the simplest one-dimensional state, self-similarity of a function $f(x)$ is $L^\alpha f(x) = f(Lx)$, where $L > 0$ is the scale length, $\alpha > 0$ some exponent representing range of amplitudes over which self-similar characteristics of $f(x)$ is displayed. The function is exactly self-similar when $\alpha=1$ [22]. In relations to fractional financial system, fractal time series also has inherent memory, i.e., supply and demand in

the future is affected by performances and characteristics in the past. Along with the self-affine properties fractal analysis leads to a new way of modelling (in)stability of financial systems [22].

3.3. Transportation Systems

Like weather and stock markets mentioned above, transportation systems are again complex and sensitive to initial conditions. The time of the day (e.g., peak hours), location (rural or urban), weather, route, etc. are all possible influential factors. The nonlinearity of road transport system arises with dynamic traffic scenarios, especially when traffic congestion occurs [29]. Hence one important application of chaos theory in transportation systems, especially in urban networks, is to predict the traffic flow [30]. This is done first by detecting and identifying chaotic characteristics in the dynamical system [31]. Kagamba and Alochukwu has listed multiple methodologies of prediction including computation of Hurst exponent, correlation dimension, Kolmogorov Entropy and maximum Lyapunov exponent [29]. Among the literatures reviewed, researchers tend to proceed with largest Lyapunov exponent (LLE). Simply saying, Lyapunov exponents measure how fast nearby trajectories or flows diverge from each other in exponential time series [32]. There are a few approaches to find the Lyapunov exponent, including an approximation of Jacobian matrices and a method evaluation accessing the progression of distances between adjacent orbits [33]. However, Cheng et al. utilized a small-data-based method [30]. This new method applies the definition of LLE directly and relates to all available data – hence assures accuracy (in small data sets). It also has the benefit of instant adaptation to alternations in parameters such as data set size, noise level, dimensions, and time delay; it is also a fast and easily implemented algorithm [32]. Let $\lambda_i, (i = 1, 2, \dots, n)$ denote the spectrum of Lyapunov exponents, the general formula to estimate λ_1 is:

$$\lambda_1(i) = \frac{1}{i\Delta t} \frac{1}{(M-i)} \sum_{j=1}^{M-i} \ln \frac{d_j(i)}{d_j(0)} \quad (12)$$

where Δt denotes sampling period, and $d_j(i)$ the separation between the j -th pair of closest neighbors after $i \cdot \Delta t$ seconds in time series. Before getting the estimation of λ_1 , there is a need to reconstruct of the phase space, hence the trajectory becomes $X = [X_1 X_2 \dots X_M]^T$. Given discrete time M , X_M is the state of the system, where $M = N - (m - 1)J$ with J as the reconstruction delay, m the embedding dimension and N representing this is an N -point time series [32].

Among the ‘traditional models’ for short-term prediction, autoregressive integrated moving average (ARIMA) model and the support vector regression (SVR) model (short-term traffic flow) and considered by Cheng et al. and Adewumi et al., respectively, are proposed as the outstanding models in each of their studies [29, 30]. SVR model, as an artificial intelligent theory-based model, has the advantage of independence of exact mathematical relationships of reponse and explanatory variable, they exhibit strong learning capabilities and reliabilities [30]. Moreover, slow rate of convergence and local optimal point are abstained in this model. SVR model is further improved by Cheng et al. by utilizing chaos theory to integrate multiple sources and measure [30]. ARIMA model is performed under the condition that the time series is free from periodic fluctuations. ARIMA model, especially SARIMA (seasonal) model, is most relevant to predicting traffic flow, especially when traffic flow has underlying seasonal pattern of peaks. However, a main drawback is the requirement of large history dataset [34, 35]. A review of urban traffic flow prediction technique in 2022 highlights the efficacy of deep learning and hybrid algorithms, surpassing traditional models like ARIMA. Graph Convolutional Networks (GCN) stand out for capturing complex spatial dependencies in road networks. Integrating GCN with RNN (recurrent neural network) addresses temporal dynamics, yet challenges remain in incorporating spatial-temporal external data to enhance predictions [35].

4. Limitations and Prospects

One significant limitation, also associated with present technology, is that predictions can only be made for short term, and going further in the future will result in increasing errors and inaccuracies.

Furthermore, the parameters that are investigated via prediction models are limited, and sometimes correlated factors are not being considered. Another drawback is that the prediction models at the present are more focused on the accuracy, but instead, stability of prediction is also of key for the varying nature of data. In general, advancement of computing technologies is at heart of future advancement of chaos theory and its applications, especially in aid of the increasing machine learning and artificial intelligence methodologies. Data collection is at heart of any forecasting problems, improvements shall focus on areas such as data collection, on the availability and accessibility on real-time data, and on the efficiency of data handling and processing.

5. Conclusion

To sum up, this study is carried out to introduce fundamental ideas in chaos theory and the applications of these in weather forecasting, financial markets, and transport systems. Chaos theory can be utilized as these systems are all complex and sensitive to change in initial conditions. In weather forecasting, the traditional model used is numerical weather prediction (NWP). Modern technique involves scale analysis and aside from statistical models there are also hybrid ones, and the machine learning / AI models, where ANN and SVM have distinct reliability. Another path proposed is to manually alter initial conditions to 'change' weather but brings along controversial debates.

Applications in stock market is related to fractal analysis with self-affinity and self-similarity characteristic of the market. Chaos theory is applied for risk management – to identify patterns of chaos. This is done by looking at the complexity of the system, and one of the metrics is the C_0 algorithm. In transport systems, one preferable methodology of prediction is largest Lyapunov exponent, which can be calculated in several ways. Models to forecast traffic flow in near future includes SVR model, ARIMA model, or by utilizing networks such as GCN. Lastly, limitations but also the future in application of chaos theory associate with computing technologies, especially with the evolving reliance on big data. One goal for the future is to be able to utilize chaos theory for stable predictions for longer terms.

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