H_∞ Estimation of Nonlinear Time-Varying Systems with Data Packet Dropouts

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Abstract. This paper handles the finite-horizon H_∞ state estimator design issue for nonlinear time-varying systems under the effect of multiplicative noise and random data packet dropouts. Stochastic nonlinear function with known statistical properties is employed to depict the nonlinear intrinsic characteristics of real systems. The phenomenon of random data packet dropouts is characterized using a random variable following the Bernoulli distribution. Multiplicative noise is incorporated as well as additive process noise. Through the utilization of stochastic analysis, matrix computation and completing squares approach, a necessary and sufficient condition is proposed to ensure the existence of ideal estimators that satisfy the H_∞ performance requirement. The time-varying gain parameters of the estimators are derived by recursively solving a group of coupled backward Riccati difference equations. Correctness of the developed estimation approach is verified via a simulation example.

Keywords: H_∞ state estimation, randomly occurring data packet dropouts, multiplicative noise, nonlinear time-varying systems, Riccati difference equations.

1. Introduction

Over the past few decades, with the development and widespread adoption of networked control systems in fields such as petroleum and petrochemical [1], medical [2], artificial intelligence [3], power grids [4], and traffic control [5], the system complexity has increased which includes nonlinearity [6]. Considering sudden environmental changes, intermittent transmission congestion, random component failures, and maintenance, the occurrence of nonlinearity often exhibits stochastic characteristics. Recently, extensive research has been conducted to address the state estimation issue of nonlinear systems. More specifically, state estimation involves utilizing known measurement information disrupted by noise to estimate the true nonlinear system state [7, 8]. For instance, in reference [8], a state estimation method has been proposed for multi-rate nonlinear systems under the FlexRay protocol. Furthermore, in real-life scenarios, due to the rapid advancement of industry, besides system nonlinearity, the time-varying characteristics has been recognized by scholars gradually as a crucial indicator reflecting changes in system parameters with time [9].

It should be noted that many system models can be established with containing multiplicative noise [10, 11]. Some characteristics of nonlinear systems can be approximated by multiplicative noise models. On the other hand, the capacity of communication networks is often limited, and during high-load network data transmission, sensor measurement data may be inevitably lost due to reasons such as channel congestion [12], signal distortion, network attacks [13], and communication delays. Incorrect handling of the phenomenon of data packet dropouts can lead to a decline in estimation performance or even failure [14, 15, 16]. In the context of nonlinear finite-horizon H_∞ state estimation, there is limited research combining packet dropouts with consideration of multiplicative noise, let alone considering time-varying parameters and providing a necessary and sufficient condition for the existence of estimators [17, 18, 19]. This constitutes the main motivation for the research in this paper.

Based on the discussions above, efforts are made in this paper to settle the H_∞ state estimation of time-varying nonlinear systems with multiplicative noise in the presence of data packet dropouts. Utilizing completing squares method, matrix computation and stochastic analysis technique, a necessary and sufficient condition is proposed to ensure the existence of estimators. Finally, a numerical simulation example is provided to demonstrate the effectiveness of this estimation approach.

2. Problem Formulation and Preliminaries

This paper addresses a discrete-time-varying stochastic nonlinear system described by a state-space model defined on a finite-horizon $k\in[0,N]$:

$$\begin{cases} x(k+1)=h(k,x(k))+\left(A(k)+\sum_{i=0}^{n}\beta_i(k)A_i(k)\right)x(k)+B(k)\omega(k), \\ y(k)=C(k)x(k)+D(k)\nu(k), \\ z(k)=L(k)x(k), \end{cases} \tag{1}$$

Where $x(k)\in\mathbb{R}^{n_x}$ is the system state vector; $h(k,x(k))$ is the stochastic nonlinear function; $\beta_i(k)\in\mathbb{R}(i=0,1,\cdots,s)$ is with $\beta_i(k)\sim\mathcal{N}(0,1)$; $y(k)\in\mathbb{R}^{n_y}$ means the measurement output; $z(k)\in\mathbb{R}^{n_z}$ represents the output of the state to be estimated; $\omega(k)\in\mathbb{R}^{n_\omega}$ and $\nu(k)\in\mathbb{R}^{n_\nu}$ are disturbance input, both of which belong to $l_2[0,N]$. $A(k),B(k),C(k),D(k)$, $A_i(k)$ and $L(k)$ are given time-varying matrices with suitable dimensions.

The stochastic nonlinear function $h(k,x(k))\,(h(k,0)=0)$ is described by the following statistical properties:

$$\begin{aligned} &E\{h(k,x(k))\,|x(k)\}=0, \\ &E\{h(j,x(j))h^T(k,x(k))\,|x(k)\}=0,k\neq j, \\ &E\{h(k,x(k))h^T(k,x(k))\,|x(k)\}\triangleq\sum_{l=1}^{q}\Theta_l(k)E\left\{x^T(k)\bar{\Gamma}_l(k)x(k)\right\}. \end{aligned} \tag{2}$$

Where $\Theta_l(k)$ and $\bar{\Gamma}_l(k)$ are known parameter matrices of appropriate dimensions, while q represents the number of independent state components.

The measurement output model with data packet dropouts is described as follows:

$$\bar{y}(k)=\varepsilon(k)y(k), \tag{3}$$

Where the random variable $\varepsilon(k)$ represents whether packet dropouts occur in the system, belonging to the Bernoulli distribution on the interval [0, 1]. That is

$$\begin{cases} \text{Prob}\{\varepsilon(k)=0\}=E\{1-\varepsilon(k)\}=1-\bar{\varepsilon}, \\ \text{Prob}\{\varepsilon(k)=1\}=E\{\varepsilon(k)\}=\bar{\varepsilon}, \\ \text{Var}\{\varepsilon(k)\}=(1-\bar{\varepsilon})\bar{\varepsilon}=\hat{\varepsilon}, \end{cases} \tag{4}$$

Where $\bar{\varepsilon} > 0$ is a given scalar, and the probability of packet dropouts is $1 - \bar{\varepsilon}$. Based on (3), we have $\bar{y}(k) = 0$ when $\varepsilon(k) = 0$ (i.e., packet dropouts appear), and $\bar{y}(k) = y(k)$ when $\varepsilon(k) = 1$ (i.e., successful data transmission occurs).

For system (1), the following time-varying estimator model is proposed:

$$\begin{cases} \hat{x}(k+1) = A(k)\hat{x}(k) + K(k)\left(\bar{y}(k) - \bar{\varepsilon}C(k)\hat{x}(k)\right), \\ \hat{z}(k) = L(k)\hat{x}(k), \end{cases} \tag{5}$$

Where $\hat{x}(k)$ and $\hat{z}(k)$ are estimates of the state $x(k)$ and the output $z(k)$, respectively; $K(k)$ is the estimator gain matrix to be designed.

Letting $e(k) \triangleq x(k) - \hat{x}(k)$ and $z_e(k) \triangleq z(k) - \hat{z}(k)$, the estimation error system can be obtained from (1), (3) and (5) as follows:

$$\begin{cases} e(k+1) = h(k,x(k)) + A(k)e(k) + \sum_{i=0}^{n} \beta_i(k)A_i(k)x(k) + B(k)\omega(k) - K(k)\tilde{\varepsilon}(k) \\ \qquad \times C(k)x(k) - K(k)\tilde{\varepsilon}(k)D(k)\nu(k) - K(k)\bar{\varepsilon}C(k)e(k) - K(k)\bar{\varepsilon}D(k)\nu(k), \\ z_e(k) = L(k)e(k). \end{cases} \tag{6}$$

In preparation for subsequent research, letting $\eta(k) \triangleq [x^T(k)\ \ e^T(k)]^T$ and $\bar{\omega}(k) \triangleq [\omega^T(k)\ \ \nu^T(k)]^T$. Based on (1) and (6), the following augmented estimation error dynamic system is obtained:

$$\begin{cases} \eta(k+1) = \bar{I}_h h\left(k, \bar{I}_e\eta(k)\right) + \bar{A}(k)\eta(k) + \sum_{i=0}^{n} \beta_i(k)\bar{A}_i(k)\eta(k) \\ \qquad + \tilde{\varepsilon}(k)\tilde{A}(k)\eta(k) + \bar{B}(k)\bar{\omega}(k) + \tilde{\varepsilon}(k)\tilde{B}(k)\bar{\omega}(k), \\ z_e(k) = \bar{L}(k)\eta(k), \end{cases} \tag{7}$$

Where

$$\bar{A}(k) \triangleq \begin{bmatrix} A(k) & 0 \\ 0 & A(k) - K(k)\bar{\varepsilon}C(k) \end{bmatrix}, \tilde{A}(k) \triangleq \begin{bmatrix} 0 & 0 \\ -K(k)C(k) & 0 \end{bmatrix}, \bar{L}(k) \triangleq [0 \quad L(k)], \bar{I}_h \triangleq \begin{bmatrix} I \\ I \end{bmatrix},$$

$$\bar{A}_i(k) \triangleq \begin{bmatrix} A_i(k) & 0 \\ A_i(k) & 0 \end{bmatrix}, \bar{B}(k) \triangleq \begin{bmatrix} B(k) & 0 \\ B(k) & -K(k)\bar{\varepsilon}D(k) \end{bmatrix}, \tilde{B}(k) \triangleq \begin{bmatrix} 0 & 0 \\ 0 & -K(k)D(k) \end{bmatrix}, \bar{I}_e \triangleq [I \quad 0].$$

In this paper, our objective is to address the finite-horizon H_∞ state estimation problem of nonlinear time-varying systems (1) with data packet dropouts and multiplicative noise. We aim to design the gain parameters $K(k)$ such that the augmented estimation error dynamic system (7) fulfills the H_∞ performance criterion below:

$$\mathcal{J} \triangleq E\left\{\sum_{k=0}^{N-1} \|z_e(k)\|^2 - \varsigma^2 \sum_{k=0}^{N-1} \|\bar{\omega}(k)\|^2\right\} - \varsigma^2\eta^T(0)M\eta(0) < 0, \tag{8}$$

Where $M > 0$ is a given weight matrix, $\varsigma > 0$ is a given disturbance attenuation level.

3. Main Results

Before presenting the main results, this chapter provides the process of the performance analysis and the design of estimator (5).

***Lemma 1*:** For the noise signal $\bar{\omega}(k)$ and initial value $\eta(0)$, considering $\eta(k)$ as the solution of system (7) over the interval [0, *N*], we can obtain:

$$\begin{aligned}\mathcal{J}_1(\eta(0),\bar{\omega}(k)) &\triangleq E\Big\{\sum\nolimits_{k=0}^{N-1}\|z_e(k)\|^2-\varsigma^2\sum\nolimits_{k=0}^{N-1}\|\bar{\omega}(k)\|^2\Big\}\\ &=E\{\eta^T(0)P(0)\eta(0)-\eta^T(N)P(N)\eta(N)\}\\ &\quad+\sum_{k=0}^{N-1}E\left\{\begin{bmatrix}\eta(k)\\ \bar{\omega}(k)\end{bmatrix}^T\begin{bmatrix}\Lambda_{11}(k+1)-P(k) & \Lambda_{12}(k+1)\\ * & -\Lambda_{22}(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \bar{\omega}(k)\end{bmatrix}\right\}.\end{aligned} \tag{9}$$

Furthermore, if $|\Lambda_{22}(k+1)|\neq 0$ for all $k\in[0,N-1]$, and by selecting $\bar{\omega}(k)=\Lambda_{22}^{-1}(k+1)\Lambda_{12}^T(k+1)\eta(k+1)$, we obtain:

$$\begin{aligned}\mathcal{J}_2(\Omega(k),\bar{\omega}(k)) &\triangleq \sum\nolimits_{k=0}^{N-1}E\{\|z_e(k)\|^2+\|\Omega(k)\|^2\}\\ &=E\{\eta^T(0)Q(0)\eta(0)-\eta^T(N)Q(N)\eta(N)\}\\ &\quad+\sum_{K=0}^{N-1}E\left\{\begin{bmatrix}\eta(k)\\ \Omega(k)\end{bmatrix}^T\begin{bmatrix}\Psi_{11}(k+1)-Q(k) & \Psi_1(k+1)\\ * & \Psi_2(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \Omega(k)\end{bmatrix}\right\}.\end{aligned} \tag{10}$$

Where $\{P(k)>0\}_{0\leqslant k\leqslant N}$ and $\{Q(k)>0\}_{0\leqslant k\leqslant N}$ are two sets of positive definite matrices, and

$$\begin{aligned}&\mathcal{A}(k)\triangleq \mathrm{diag}\{A(k),A(k)\},\ \Omega(k)\triangleq \bar{K}(k)\mathcal{C}(k)\eta(k),\bar{K}(k)\triangleq[0\quad K^T(k)]^T,\\ &\Lambda_{12}(k+1)\triangleq \bar{A}^T(k)P(k+1)\bar{B}(k),\Delta(k+1)\triangleq\Lambda_{22}^{-1}(k+1)\Lambda_{12}^T(k+1),\\ &\Lambda_{11}(k+1)\triangleq \mathrm{tr}\Big[\bar{I}_h^TP(k+1)\bar{I}_h\bar{\Gamma}_l(k)\Big]\bar{I}_e^T\sum\nolimits_{l=1}^{q}\Theta_l(k)\bar{I}_e+\bar{L}^T(k)\bar{L}(k)+\bar{A}^T(k)\\ &\qquad\times P(k+1)\bar{A}(k)+\sum\nolimits_{i=0}^{n}A_i^T(k)P(k+1)A_i(k)+\hat{\varepsilon}\tilde{A}_k^TP(k+1)\tilde{A}_k,\\ &\Lambda_{22}(k+1)\triangleq\varsigma^2I-\bar{B}^T(k)P(k+1)\bar{B}(k)-\hat{\varepsilon}\tilde{B}^T(k)P(k+1)\tilde{B}(k),\\ &\Psi(k+1)\triangleq \mathrm{tr}\Big[\bar{I}_h^TP(k+1)\bar{I}_h\bar{\Gamma}_l(k)\Big]\bar{I}_e^T\sum\nolimits_{l=1}^{q}\Theta_l(k)\bar{I}_e+\Big(\mathcal{A}(k)+\bar{B}(k)\Delta(k+1)\Big)^T\\ &\qquad\times Q(k+1)\Big(\mathcal{A}(k)+\bar{B}(k)\Delta(k+1)\Big)+\hat{\varepsilon}\Big(\tilde{A}(k)+\tilde{B}(k)\Delta(k+1)\Big)^T\\ &\qquad\times Q(k+1)\Big(\tilde{A}(k)+\tilde{B}(k)\Delta(k+1)\Big)+\sum\nolimits_{i=0}^{n}A_i^T(k)Q(k+1)A_i(k),\\ &\Psi_{11}(k+1)\triangleq\Psi(k+1)+\bar{L}^T(k)\bar{L}(k)-Q(k),\Psi_2(k+1)\triangleq Q(k+1)+I,\\ &\Psi_1(k+1)\triangleq\Big(\mathcal{A}(k)+\bar{B}(k)\Delta(k+1)\Big)^TQ(k+1),\ \mathcal{C}(k)\triangleq[0\quad -\bar{\varepsilon}C(k)].\end{aligned} \tag{11}$$

Proof: Letting $\bar{V}(k)\triangleq\eta^T(k+1)P(k+1)\eta(k+1)-\eta^T(k)P(k)\eta(k)$, according to the trajectory of system (7), we can obtain:

$$
\begin{aligned}
E\left\{\bar{V}(k)\right\} = & E\left\{h^T\left(k,\bar{I}_e\eta(k)\right)\bar{I}_h^T P(k+1)\bar{I}_h h\left(k,\bar{I}_e\eta(k)\right) + \eta^T(k)\bar{A}^T(k)\right. \\
& \times P(k+1)\bar{A}(k)\eta(k) + \sum\nolimits_{i=0}^{n}\beta_i^2(k)\eta^T(k)\bar{A}_i^T(k)P(k+1)\bar{A}_i(k)\eta(k) \\
& + \hat{\varepsilon}\eta^T(k)\tilde{A}^T(k)P(k+1)\tilde{A}(k)\eta(k) + \bar{\omega}^T(k)\bar{B}^T(k)P(k+1)\bar{B}(k)\bar{\omega}(k) \\
& + \bar{\omega}^T(k)\bar{B}(k)P(k+1)\bar{A}(k)\eta(k) + \hat{\varepsilon}\bar{\omega}^T(k)\tilde{B}^T(k)P(k+1)\tilde{B}(k)\bar{\omega}(k) \\
& \left. + \eta^T(k)\bar{A}^T(k)P(k+1)\bar{B}(k)\bar{\omega}(k) + \eta^T(k)P(k)\eta(k)\right\}.
\end{aligned} \tag{12}
$$

Given the statistical property (2) of nonlinearity and the properties of matrix trace, it can be derived that:

$$
\begin{aligned}
& E\left\{h^T\left(k,\bar{I}_e\eta(k)\right)\bar{I}_h^T P(k+1)\bar{I}_h h\left(k,\bar{I}_e\eta(k)\right)\right\} \\
= & E\left\{\mathrm{tr}\left[h^T\left(k,\bar{I}_e\eta(k)\right)\bar{I}_h^T P(k+1)\bar{I}_h h\left(k,\bar{I}_e\eta(k)\right)\right]\right\} \\
= & E\left\{\eta^T(k)\bar{I}_e^T\sum\nolimits_{l=1}^{q} tr\left[\bar{I}_h^T P(k+1)\bar{I}_h\bar{\Gamma}_l(k)\right]\Theta_l(k)\bar{I}_e\eta(k)\right\}.
\end{aligned} \tag{13}
$$

According to (12) and (13), combining $\sum\nolimits_{k=0}^{N-1}E\{\|z_e(k)\|^2\}$ with the zero term $\sum\nolimits_{k=0}^{N-1}E$ $\left\{\bar{V}(k)-\bar{V}(k)\right\}$ and $\varsigma^2\sum\nolimits_{k=0}^{N-1}E\{\|\bar{\omega}(k)\|^2-\|\bar{\omega}(k)\|^2\}$, we can derive that:

$$
\begin{aligned}
& \sum\nolimits_{k=0}^{N-1}E\{\|z_e(k)\|^2\} \\
= & \varsigma^2\sum\nolimits_{k=0}^{N-1}E\{\|\bar{\omega}(k)\|^2\} + E\{\eta^T(0)P(0)\eta(0) - \eta^T(N)P(N)\eta(N)\} \\
& + \sum_{k=0}^{N-1}E\left\{\begin{bmatrix}\eta(k)\\ \bar{\omega}(k)\end{bmatrix}^T\begin{bmatrix}\Lambda_{11}(k+1)-P(k) & \Lambda_{12}(k+1)\\ * & -\Lambda_{22}(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \bar{\omega}(k)\end{bmatrix}\right\}.
\end{aligned} \tag{14}
$$

Similarly, taking $\Omega(k)\triangleq\bar{K}(k)\mathcal{C}(k)\eta(k)$ into account, we obtain:

$$
\bar{A}(k)\eta(k) = \mathcal{A}(k)\eta(k) + \Omega(k). \tag{15}
$$

If $|\Lambda_{22}(k+1)|\neq 0$ for all $k\in[0,N-1]$, and by selecting $\bar{\omega}(k)=\Lambda_{22}^{-1}(k+1)\Lambda_{12}^T(k+1)\eta(k+1)$, we obtain:

$$
\begin{aligned}
& \sum\nolimits_{k=0}^{N-1}E\{\|z_e(k)\|^2\} \\
= & E\{\eta^T(0)Q(0)\eta(0) - \eta^T(N)Q(N)\eta(N)\} - \sum\nolimits_{k=0}^{N-1}E\{\|\Omega(k)\|^2\} \\
& + \sum_{k=0}^{N-1}E\left\{\begin{bmatrix}\eta(k)\\ \Omega(k)\end{bmatrix}^T\begin{bmatrix}\Psi_{11}(k+1)-Q(k) & \Psi_1(k+1)\\ * & \Psi_2(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \Omega(k)\end{bmatrix}\right\}.
\end{aligned} \tag{16}
$$

It is readily apparent that equations (9) and (10) are respectively ensured by (14) and (16).

Next, we will establish a necessary and sufficient condition for designing a time-varying state estimator (5) under the specified performance constraint (8) using the method of completing squares and proof by contradiction.

***Lemma 2*:** For the nonlinear time-varying system (1)-(3), given the disturbance attenuation level $\varsigma>0$ and a matrix $M>0$, considering system (7) for all non-zero $\{\bar{\omega}(k)\}_{0\leqslant k\leqslant N-1}$, the performance requirement (8) is fulfilled if and only if there exists a set of real-valued matrices

$\{P(k)>0\}_{0\leqslant k\leqslant N-1}$ (with the final condition $P(N)=0$) ensuring the following backward recursive Riccati difference equation:

$$\Lambda_{11}(k+1)+\Lambda_{12}(k+1)\Lambda_{22}^{-1}(k+1)\Lambda_{12}^{T}(k+1)=P(k) \tag{17}$$

Holds with

$$\Lambda_{22}(k+1)>0 \text{ and } P(0)<\varsigma^{2}M. \tag{18}$$

Proof: Sufficiency: For matrices $\{P(k)>0\}_{0\leqslant k\leqslant N}$ in (17), viewing (14), we obtain:

$$\begin{aligned}&\sum\nolimits_{k=0}^{N-1}E\{\|z_{e}(k)\|^{2}\}-\varsigma^{2}\sum\nolimits_{k=0}^{N-1}E\{\|\bar{\omega}(k)\|^{2}\}\\ &=E\{\eta^{T}(0)P(0)\eta(0)-\eta^{T}(N)P(N)\eta(N)\}+\sum\nolimits_{k=0}^{N-1}E\{\eta^{T}(k)\left(\Lambda_{11}(k+1)-P(k)\right)\\ &\quad\times\eta(k)+2\eta^{T}(k)\Lambda_{12}(k+1)\bar{\omega}(k)-\bar{\omega}^{T}(k)\Lambda_{22}(k+1)\bar{\omega}(k)\}\\ &=E\{\eta^{T}(0)P(0)\eta(0)-\eta^{T}(N)P(N)\eta(N)\}\\ &\quad+\sum\nolimits_{k=0}^{N-1}E\{-\left(\bar{\omega}(k)-\bar{\omega}^{*}(k)\right)^{T}\Lambda_{22}(k+1)\left(\bar{\omega}(k)-\bar{\omega}^{*}(k)\right)\},\end{aligned} \tag{19}$$

Where $\bar{\omega}^{*}(k)\triangleq\Lambda_{22}^{-1}(k+1)\Lambda_{12}^{T}(k+1)\eta(k)$.

Because of $\Lambda_{22}(k+1)>0$ and $P(0)<\varsigma^{2}M$ for all nonzero $\{\bar{\omega}(k)\}_{0\leqslant k\leqslant N}$, we can get from (17) and $P(N)=0$ that

$$\begin{aligned}\mathcal{J}&<\sum\nolimits_{k=0}^{N-1}E\{\|z_{e}(k)\|^{2}\}-\varsigma^{2}\sum\nolimits_{k=0}^{N-1}E\{\|\bar{\omega}(k)\|^{2}\}-\eta^{T}(0)P(0)\eta(0)\\ &=-\sum\nolimits_{k=0}^{N-1}E\{\left(\bar{\omega}(k)-\bar{\omega}^{*}(k)\right)^{T}\Lambda_{22}(k+1)\left(\bar{\omega}(k)-\bar{\omega}^{*}(k)\right)\}<0,\end{aligned} \tag{20}$$

Which corresponds to (8).

The proof of necessity has similarity with that in *Lemma 2* of [20], which is omitted here. The proof is now complete.

Building upon *Lemma 2*, we are prepared to present the design algorithm for the estimator (5) and determine the gain matrix $K(k)$ of the estimator.

Theorem 1: Considering the aforementioned nonlinear system (1)-(3), for a specific disturbance attenuation level $\varsigma>0$ and a weight matrix $M>0$, the estimator (5) fulfills the performance constraint (8) for all

Non-zero $\{\bar{\omega}(k)\}_{0\leqslant k\leqslant N-1}$ if (17) and the following recursive Riccati difference equation

$$\Psi(k+1)+\bar{L}^{T}(k)\bar{L}(k)-\Psi_{1}(k+1)\Psi_{2}^{-1}(k+1)\Psi_{1}^{T}(k+1)=Q(k) \tag{21}$$

Have solutions $\{P(k),Q(k),K(k)\}_{0\leqslant k\leqslant N-1}$ and satisfy

$$P(N)=Q(N)=0, \tag{22}$$

$$\Psi_{2}(k+1)>0,\Lambda_{22}(k+1)>0,P(0)<\varsigma^{2}M, \tag{23}$$

$$\bar{K}^{*}(k)=\arg\min_{\bar{K}(s)}\left\|\bar{K}(k)\mathcal{C}(k)+\Psi_{2}^{-1}(k+1)\Psi_{1}^{T}(k+1)\right\|_{F} \tag{24}$$

Where $\|\cdot\|_F$ is the Frobenius norm, and other matrix parameters are detailed in *Lemma 1*.

Proof: Firstly, if $\{P(k)\}_{0\leqslant k\leqslant N-1}$ exist which satisfy (17) and (23), it can be readily deduced from *Lemma 2* that the performance criterion (8) is satisfied by the estimation error dynamic system (7). The noise in the worst-case scenario can be represented as $\bar{\omega}^{*}(k)\triangleq\Lambda_{22}^{-1}(k+1)\Lambda_{12}^{T}(k+1)\eta(k)$.

By employing the method of completing squares and considering the worst-case noise, it can be derived from *Lemma 1* that:

$$\begin{aligned}
&\mathcal{J}_2(\Omega(k),\bar{\omega}(k)) \\
=&E\{\eta^T(0)Q(0)\eta(0)-\eta^T(N)Q(N)\eta(N)\} \\
&+\sum_{k=0}^{N-1}E\{\eta^T(k)\Big(\Psi(k+1)+\bar{L}^T(k)\bar{L}(k)-Q(k)-\Psi_1(k+1)\Psi_2^{-1}(k+1) \\
&\times\Psi_1^T(k+1)\big)\eta(k)\big)+(\Omega(k)-\Omega^*(k))^T\Psi_2(k+1)(\Omega(k)-\Omega^*(k))\} \\
\leqslant&E\{\eta^T(0)Q(0)\eta(0)-\eta^T(N)Q(N)\eta(N)\} \\
&+\sum_{k=0}^{N-1}E\{\eta^T(k)\Big(\Psi(k+1)+\bar{L}^T(k)\bar{L}(k)-Q(k)-\Psi_1(k+1)\Psi_2^{-1}(k+1) \\
&\times\Psi_1^T(k+1)\big)\eta(k)\big)+\left\|\breve{L}^T\bar{K}(k)\mathcal{C}(k)+\Psi_2^{-1}(k+1)\Psi_1^T(k+1)\right\|_F^2 \\
&\times\|\Psi_2(k+1)\|_F\|\eta(k)\|^2\},
\end{aligned} \tag{25}$$

Where $\Omega^*(k)\triangleq\Psi_2^{-1}(k+1)\Psi_1^T(k+1)\eta(k)$. Additionally, this part also demonstrates that the gain parameter $K(k)$ satisfies (2.21) and (2.24). The proof is complete.

Theorem 2: Set the disturbance rejection level $\varsigma>0$, matrix $M>0$ and the constants $\delta(k)>0,\tau(k)>0$. The system (7) fulfills the H_∞ performance requirement (8) for all nonzero $\{\bar{\omega}(k)\}_{0\leqslant k\leqslant N-1}$ if there is a sequence of solutions $\{P(k),Q(k),K(k)\}_{0\leqslant k\leqslant N-1}$ to the following recursive Riccati difference equation:

$$\Lambda_{11}(k+1)+\bar{\Lambda}_{12}(k+1)\bar{\Lambda}_{22}^{-1}(k+1)\bar{\Lambda}_{12}^T(k+1)=P(k) \tag{26}$$

$$\bar{\Psi}(k+1)+\bar{L}^T(k)\bar{L}(k)-\bar{\Psi}_1(k+1)\psi_2^{-1}(k+1)\bar{\Psi}_1^T(k+1)=Q(k) \tag{27}$$

With

$$P(N)=Q(N)=0, \tag{28}$$

$$\Psi_2(k+1)>0,\ \bar{\Lambda}_{22}(k+1)>0,\ P(0)<\varsigma^2M, \tag{29}$$

$$\bar{K}^*(k)=Z_1^\dagger(k+1)Z_2(k+1)\mathcal{C}^\dagger(k), \tag{30}$$

$$Y(k)\leqslant\tau(k)I, \tag{31}$$

Were

$$\hat{B}(k)\triangleq\begin{bmatrix}\bar{B}_1(k) & \bar{B}_2(k)\end{bmatrix},\breve{B}(k)\triangleq\begin{bmatrix}0 & \tilde{B}_2(k)\end{bmatrix},\bar{S}\triangleq[0\ \ I],U\triangleq[I\ \ 0],$$

$$\bar{B}_1(k)\triangleq\begin{bmatrix}B_k & 0\\ B_k & 0\end{bmatrix},\bar{B}_2(k)\triangleq\begin{bmatrix}0 & 0\\ 0 & -\bar{\varepsilon}\delta^{-1}(k)I\end{bmatrix},\tilde{B}_2(k)\triangleq\begin{bmatrix}0 & 0\\ 0 & -\delta^{-1}(k)I\end{bmatrix},$$

$$\begin{aligned}
&\bar{\Lambda}_{22}(k+1) \triangleq \varsigma^2 I - \hat{B}^T(k)P(k+1)\hat{B}(k) - \hat{\varepsilon}\breve{B}^T(k)P(k+1)\breve{B}(k) - \tau(k)U^TU,\\
&\bar{\Delta}(k+1) \triangleq \bar{\Lambda}_{22}^{-1}(k+1)\bar{\Lambda}_{12}^T(k+1), Y(k) \triangleq \varsigma^2\delta^2(k)\bar{S}^TD^T(k)K^T(k)K(k)D(k)\bar{S},\\
&\bar{\Psi}_1(k+1) \triangleq \left(\mathcal{A}(k) + \hat{B}(k)\bar{\Delta}(k+1)\right)^TQ(k+1),\\
&\bar{\Psi}(k+1) \triangleq \mathrm{tr}\left[\bar{I}_h^TP(k+1)\bar{I}_h\bar{\Gamma}_l(k)\right]\bar{I}_e^T\sum\nolimits_{l=1}^{q}\Theta_l(k)\bar{I}_e\\
&\qquad + \left(\mathcal{A}(k) + \hat{B}(k)\bar{\Delta}(k+1)\right)^TQ(k+1)\left(\mathcal{A}(k) + \hat{B}(k)\bar{\Delta}(k+1)\right)\\
&\qquad + \hat{\varepsilon}\left(\tilde{A}(k) + \breve{B}(k)\bar{\Delta}(k+1)\right)^TQ(k+1)\left(\tilde{A}(k) + \breve{B}(k)\bar{\Delta}(k+1)\right)\\
&\qquad + \sum\nolimits_{i=0}^{n}A_i^T(k)Q(k+1)A_i(k),\ \bar{\Lambda}_{12}(k+1) \triangleq \bar{A}^T(k)P(k+1)\hat{B}(k),\\
&Z_2(k+1) \triangleq -\Psi_2^{-1}(k)Q(k+1)\left(I + \hat{B}(k)\bar{\Lambda}_{22}^{-1}(k+1)\hat{B}^T(k)P(k+1)\right)\mathcal{A}(k),\\
&Z_1(k+1) \triangleq I + \Psi_2^{-1}(k)Q(k+1)\hat{B}(k)\bar{\Lambda}_{22}^{-1}(k+1)\hat{B}^T(k)P(k+1).
\end{aligned} \tag{32}$$

Proof: Choosing $\bar{\nu}(k) \triangleq [0 \quad (\delta(k)K(k)D(k)\nu(k))^T]^T$, where $\delta(k) > 0$ is used to provide additional degrees of freedom to the system during estimation, and then selecting $\tilde{\omega}(k) \triangleq [\bar{\omega}^T(k) \quad \bar{\nu}^T(k)]^T$, we rewrite system (7) as follows:

$$\begin{cases}\eta(k+1) = \bar{I}_hh\left(k, \bar{I}_e\eta(k)\right) + \bar{A}(k)\eta(k) + \sum\nolimits_{i=0}^{n}\beta_i(k)\bar{A}_i(k)\eta(k)\\ \qquad\qquad + \tilde{\varepsilon}(k)\tilde{A}(k)\eta(k) + \hat{B}(k)\tilde{\omega}(k) + \tilde{\varepsilon}(k)\breve{B}(k)\tilde{\omega}(k),\\ z_e(k) = \bar{L}(k)\eta(k).\end{cases} \tag{33}$$

In terms of Lemma 1, we observe that (30) constitutes an optimal solution to the following problem:

$$\min_{\bar{K}(k)}\left\|Z_1(k+1)\bar{K}(k)\mathcal{C}(k) - Z_2(k+1)\right\|_F, \tag{34}$$

Which can be further expressed as:

$$\min_{\bar{K}(k)}\left\|\bar{K}(k)\mathcal{C}(k) + \Psi_2^{-1}(k+1)\bar{\Psi}_1^T(k+1)\right\|_F. \tag{35}$$

By utilizing (14) and Theorem 1, presuming that there is a set of solutions for the recursive Riccati difference equations (26) and (27) under the conditions of equations (28)-(31), we obtain:

$$\begin{aligned}
&\sum\nolimits_{k=0}^{N-1}E\left\{\|z_e(k)\|^2\right\}\\
&= E\{\eta^T(0)P(0)\eta(0) - \eta^T(N)P(N)\eta(N)\}\\
&\quad + \varsigma^2\sum\nolimits_{k=0}^{N-1}E\left\{\|\bar{\omega}(k)\|^2 + \|\bar{\nu}(k)\|^2\right\} + \sum\nolimits_{k=0}^{N-1}E\left\{\eta^T(k)\left(\Lambda_{11}(k+1) - P(k)\right.\right.\\
&\quad \left. + \bar{\Lambda}_{12}(k+1)\bar{\Lambda}_{22}^{-1}(k+1)\bar{\Lambda}_{12}^T(k+1)\right)\eta(k) - (\tilde{\omega}(k) - \tilde{\omega}^*(k))^T\bar{\Lambda}_{22}(k+1)\\
&\times(\tilde{\omega}(k) - \tilde{\omega}^*(k))\} - \sum\nolimits_{k=0}^{N-1}E\left\{\tau(k)(U\tilde{\omega}(k))^T(U\tilde{\omega}(k))\right\}\\
&< \varsigma^2\eta^T(0)M\eta(0) + \varsigma^2\sum\nolimits_{k=0}^{N-1}E\left\{\|\bar{\omega}(k)\|^2\right\} + \sum\nolimits_{k=0}^{N-1}E\left\{\bar{\omega}^T(k)\left(Y(k) - \tau(k)I\right)\bar{\omega}(k)\right\},
\end{aligned} \tag{36}$$

Where $\overline{\omega}^*(k) \triangleq \overline{\Lambda}_{22}^{-1}(k+1)\overline{\Lambda}_{12}^T(k+1)\eta(k)$.

In combination with (31), we derive from equation (36) that:

$$\sum_{k=0}^{N-1} E\{\|z_e(k)\|^2\} < \varsigma^2 \sum_{k=0}^{N-1} E\{\|\overline{\omega}(k)\|^2\} + \varsigma^2\eta^T(0)M\eta(0). \tag{37}$$

Finally, we deduce that the time-varying estimator (5) ensures that system (7) satisfies the H_∞ performance criterion (8). The proof is complete.

4. Simulation Results

Considering a nonlinear time-varying stochastic system (1) over the finite time horizon $k \in [0, 100]$, and a set of parameters are used as follows:

$$A(k) = \begin{bmatrix} -0.02\sin(5k) & 0.02 \\ 0.04 & 0.04 \end{bmatrix}, C(k) = \begin{bmatrix} 0.4 & 0.1\sin(5k) \\ 0.5 & -0.3 \end{bmatrix}, \varsigma = 1,$$

$$B(k) = \begin{bmatrix} 0.2\sin(3k) \\ 0.4 \end{bmatrix}, D(k) = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, L(k) = \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \end{bmatrix}, M = 2I.$$

Choose the following stochastic nonlinear function:

$$h(k,x(k)) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}(0.2x_1(k)\epsilon_1(k) + 0.3x_2(k)\epsilon_2(k))$$

Where $x_\Im(k)\,(\Im = 1,2)$ stands for the $\Im$th element of $x(k)$, and $\epsilon_{\mathscr{H}}(k) \sim \mathcal{N}(0,1)\,(\mathscr{H} = 1,2)$ are uncorrelated Gaussian white noises. $h(k,x(k))$ fulfills

$$\mathbb{E}\{h(k,x(k))|x(k)\} = 0,$$

$$\mathbb{E}\{h(k,x(k))h^T(k,x(k))|x(k)\} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^T \mathbb{E}\left\{x^T(k)\begin{bmatrix} 0.04 & 0 \\ 0 & 0.09 \end{bmatrix}x(k)\right\}.$$

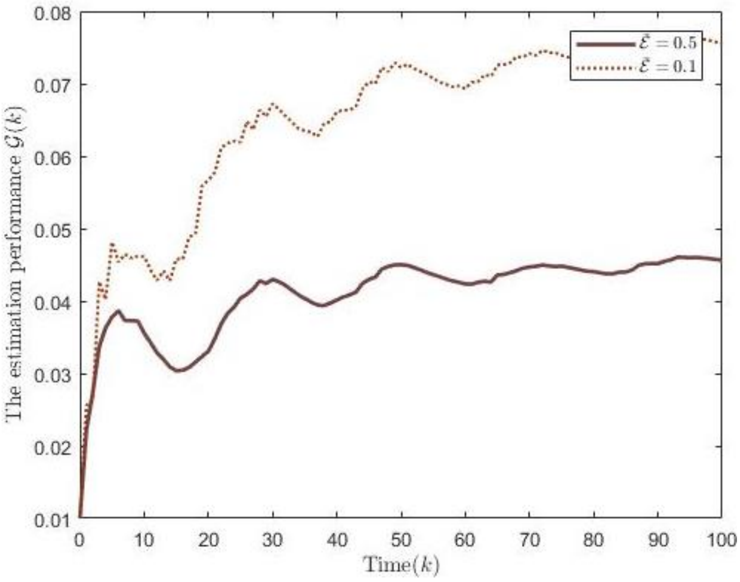

Figure 1. The estimation performance $\mathcal{G}(k)$ for different $\overline{\varepsilon}$

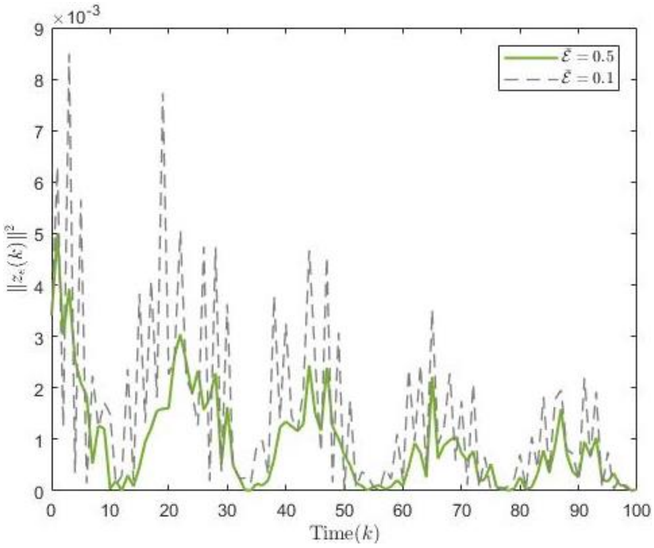

Figure 2. The norm of estimation error $\|z_e(k)\|^2$ for different $\bar{\varepsilon}$

We calculate the time-varying parameter matrices of the estimator (5) by defining $\delta(k)=2.5$ and $\tau(k)=0.25$, which are listed in Table 1. The results of Theorem 2 encompass all crucial factors of nonlinear systems, such as time-varying parameters matrices, info on multiplicative noise, and the probability of packet dropouts. By solving the backward recursive Riccati difference equations and expressing the gain of the time-varying estimator as (30) using the Moore-Penrose pseudoinverse in Theorem 2, the estimator (5) is designed.

Table 1. Time-varying gain parameters of the state estimator (5)

	$k=1$	$k=2$	$k=3$	$k=4$
$K(k)$	$\begin{bmatrix}0.1022 & 0.0637\\ 0.0469 & 0.0171\end{bmatrix}$	$\begin{bmatrix}-0.9066 & 0.038\\ -0.2567 & 0.0098\end{bmatrix}$	$\begin{bmatrix}0.0968 & 0.0614\\ 0.0493 & 0.0159\end{bmatrix}$	$\cdots$

The initial values of the system state and the estimator state are set to $x(0)=[0.2 \quad 0.3]^T$ and $\hat{x}(0)=[0.1 \quad 0.2]^T$, respectively. Noise signals are $\omega(k)=0.2\cos(2k)e^{-0.01k}$ and $\nu(k)=0.2\sin(3k)e^{-0.01k}$. The simulation curves are depicted in Figs. 1-2. Fig. 1 illustrates the H_∞ estimation performance $\mathcal{G}(k)$ under different $\bar{\varepsilon}$ with

$$\mathcal{G}(k)=\sqrt{\frac{\Sigma_{\iota=0}^{k}E\{\|z_e(\iota)\|^2\}}{\Sigma_{\iota=0}^{k}(\|\bar{\omega}(\iota)\|^2)+\eta^T(0)M\eta(0)}}<\varsigma(k=0,1,...,N-1),$$

From Fig 1, we know that $\mathcal{G}(k)<\varsigma^2$, which means that the developed estimation algorithm is capable of satisfying the H_∞ performance constraint (8). Furthermore, it is evident that as $\bar{\varepsilon}$ decreases (i.e., the probability of successful data transmission decreases), the worse the H_∞ estimation performance becomes. Fig 2 illustrates the norm of the estimation error (i.e., $\|z_e(k)\|^2$) under different $\bar{\varepsilon}$, similarly, it is shown that the smaller the value of $\bar{\varepsilon}$, the bigger the norm of the estimation error, the worse the estimation performance is.

5. Conclusion

This paper has tackled the finite-horizon H_∞ state estimation problem for stochastic nonlinear time-varying systems subject to data packet dropouts and multiplicative noise. The random packet dropouts have been characterized by a random variable following a Bernoulli distribution. In addition to additive process noise, random multiplicative noise has been incorporated into the system model. A necessary and sufficient condition for the existence of an ideal estimator guaranteeing the satisfaction of the H_∞ performance constraint has been provided, employing stochastic analysis, matrix computation and completing squares method. Time-varying gain parameters for the estimator have been obtained by solving a set of coupled backward recursive Riccati difference equations and computing the corresponding Moore-Penrose pseudo-inverse. Throughout this process, detailed derivation and proof have been presented, along with a numerical simulation example to indicate the availability of the proposed estimation algorithm. Future research endeavors may encompass 1) investigations into estimation problems of nonlinear systems with input delays, and 2) exploration into estimation problems involving uncertainty in probability of packet dropouts [21, 22].

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References

[1] Dongmei Wang, Shaoxiong Shi, Jingyi Lu, et al. Research on gas pipeline leakage model identification driven by digital twin. Systems Science & Control Engineering, 2023, 11 (1): 2180687.

[2] Xiang Li, Minglei Li, Pengfei Yan, et al. Deep learning attention mechanism in medical image analysis: Basics and beyonds. International Journal of Network Dynamics and Intelligence, 2023: 93-116.

[3] Lin Xu, Jiaqiang Du, Baoye Song, et al. A combined backstepping and fractional-order PID controller to trajectory tracking of mobile robots. Systems Science & Control Engineering, 2022, 10 (1): 134-141.

[4] Valery Rubí Rosales-Valladares, Nadia Maria Salgado-Herrera, Osvaldo Rodríguez-Hernández, et al. Power hardware in the loop methodology applied in the integration of wind energy conversion system under fluctuations: A case study. Energy Sources, Part A: Recovery, Utilization, and Environmental Effects, 2024, 46 (1): 2767-2791.

[5] Fuya Yuan, Huijun Sun, Liujiang Kang, et al. Passenger flow control strategies for urban rail transit networks. Applied Mathematical Modelling, 2020, 82: 168-188.

[6] Yi Shui, Lu Dong, Ya Zhang, et al. Event-based adaptive fuzzy tracking control for nonlinear systems with input magnitude and rate saturations. International Journal of Systems Science, 2023, 54 (16): 3045-3058.

[7] Yue Luo, Jiayu Zhou, Wen Yang. Distributed state estimation with colored noises. IEEE Transactions on Circuits and Systems II: Express Briefs, 2021, 69 (6): 2807-2811.

[8] Yuxuan Shen, Zidong Wang, Hongli Dong, et al. Distributed recursive state estimation for a class of multi-rate nonlinear systems over wireless sensor networks under FlexRay protocols. IEEE Transactions on Network Science and Engineering, 2022, 10 (3): 1551-1563.

[9] Bo Shen, Zidong Wang, Dong Wang, et al. State-saturated recursive filter design for stochastic time-varying nonlinear complex networks under deception attacks. IEEE Transactions on Neural Networks and Learning Systems, 2019, 31 (10): 3788-3800.

[10] Hang Geng, Zidong Wang, Yan Liang, et al. Tobit Kalman filter with time-correlated multiplicative sensor noises under redundant channel transmission. IEEE Sensors Journal, 2017, 17 (24): 8367-8377.

[11] Lei Guo, Wenshuo Li, Yukai Zhu, et al. Composite disturbance filtering: A novel state estimation scheme for systems with multi-source, Heterogeneous, and Isomeric Disturbances. IEEE Open Journal of the Industrial Electronics Society, 2023, 4: 387-400.

[12] Wen Li, Yugang Niu, Zhiru Cao. Event-triggered sliding mode control for multi-agent systems subject to channel fading. International Journal of Systems Science, 2022, 53 (6): 1233-1244.

[13] Huimin Tao, Hailong Tan, Qiwen Chen, et al. H_∞ state estimation for memristive neural networks with randomly occurring DoS attacks. Systems Science & Control Engineering, 2022, 10 (1): 154-165.

[14] Xianming Zhang, Qinglong Han, Xiaohua Ge. A novel approach to H_∞ performance analysis of discrete-time networked systems subject to network-induced delays and malicious packet dropouts. Automatica, 2022, 136: 110010.

[15] Langwen Zhang, Wei Xie, Jinfeng Liu. Robust control of saturating systems with Markovian packet dropouts under distributed MPC. ISA Transactions, 2019, 85: 49-59.

[16] Hong Lin, James Lam, Zidong Wang, et al. State estimation over non-acknowledgment networks with Markovian packet dropouts. Automatica, 2019, 109: 108484.

[17] Nan Hou, Jiahui Li, Hongjian Liu, et al. Finite-horizon resilient state estimation for complex networks with integral measurements from partial nodes. Science China Information Sciences, 2022, 65 (3): 132205.

[18] Kewang Huang, Feng Pan. Finite-horizon H_∞ control for time-varying state-saturated systems under stochastic communication protocol. Mathematical Problems in Engineering, 2021, 2021: 1-11.

[19] Hongli Dong, Nan Hou, Zidong Wang. Fault estimation for complex networks with randomly varying topologies and stochastic inner couplings. Automatica, 2020, 112: 108734.

[20] Zongjie Luo, Yanqin Wang, Dongyan Dai, et al. Finite-horizon security-guaranteed non-fragile H_∞ estimation under integral measurements. The 3rd Conference on Fully Actuated System Theory and Applications (FASTA2024), Shenzhen, China, 10th-12th May, 2024, accepted on Mar. 18th, 2024.

[21] Chunyan Han, Zidong Wang, Huanshui Zhang, et al. Stabilization and optimal control for discrete-time Markov jump linear system with multiplicative noises and input delays: A complete solution. IEEE Transactions on Automatic Control, 2023.

[22] Weilu Chen, Jun Hu, Xiaoyang Yu, et al. Protocol-based fault detection for discrete delayed systems with missing measurements: the uncertain missing probability case. IEEE Access, 2018, 6: 76616-76626.