

Insight Of the Quantized Gravity: From Harmonic Oscillator to Black Hole

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Abstract. In the long history of physics development, harmonic oscillator is always a very important tool. This is because of its beauty in not only the motion but also the relevant mathematic form. This paper states the history related to harmonic oscillators, and explores its probable application in quantized gravity and the quantum Banados-Teitelboim-Zanelli (qBTZ) black hole especially. This paper starts from the harmonic oscillators in classical mechanics, and then stresses the introduction of creation and annihilation operators in quantum harmonic oscillators. This indicates the importance of the idea that harmonic oscillator system can be described by annihilation (creation) operator. In addition, before expanding the idea that depict the system by annihilation(creation) operator on gravity, this paper shows a brief introduction on quantum field theory. After that, the quantized gravity will be mentioned, where this paper deems black hole as an important system for people to study the essence of gravity. Finally, the qBTZ black hole is introduced as an example.

Keywords: Harmonic oscillator; Quantum field theory; Quantized gravity; Black hole.

1. Introduction

The harmonic oscillators play vital roles in many branches of modern physics. The paper delves into the intricacies of harmonic oscillators, beginning with an exploration of the classical oscillator, whose behavior is elucidated by Newton's second law. It then seamlessly transitions into the quantum realm. Particularly, the method employed, which involves the introduction of annihilation and creation operators, is known as canonical quantization. This approach holds significant relevance across various domains of physics, prominently in quantum field theory (QFT) and many-body theory. Throughout this paper, the intersections of these concepts with QFT and many-body physics are duly acknowledged and discussed. In this article, it is stressed that physicists always cope with harmonic oscillators in different conditions. In classical mechanics, it is a spring attached to a ball, and in quantum field theory it is explained by the creation and annihilation operators, and further in quantum field theory, oscillator becomes a set of infinite oscillators, to depict the nature of the field [1].

The simplest case is the harmonic oscillators in classical mechanics. It is well-known that the classical system can be described by the Newton's law, $\frac{\partial^2}{\partial t^2}x + \omega^2x = 0$. Setting the balance point as the coordinate origin point, then the differential equation's solution can be written as $Ae^{i\omega t} + Be^{-i\omega t}$, where A, B is a complex constant. Thus, one can identify that the solution is just a trigonometric function in a real explanation [2]. The paper does not focus on classical case, and more information is in almost every classical mechanic textbook. However, in quantum mechanics, harmonic oscillators problem can be solved by solving the Schrödinger equation with the Hamiltonian, $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$. It has the same form with classical mechanics if dismissing the physical quantities have been quantized. The most significant question is solving the time-independent Schrödinger equation $H\psi = E\psi$. This paper does not focus on finding the solution of this equation but focus on the idea of annihilation and creation operators. In addition, this paper expands the idea to the quantum field theory, in which the fields are described by creation and annihilation of particles, or the particles is excited states of the fields on the other hand.

Finally, this paper contemplates the prospect of quantized gravity. Given the absence of empirical evidence, no universally recognized theory of quantized gravity exists to date. Consequently, researchers persist in their endeavors, despite the inability to validate any proposed theories through

experimentation due to the lack of a suitable observational framework. Therefore, thought experiments play a pivotal role in advancing the exploration of quantized gravity. Among these, black holes emerge as one of the most prominent physical systems, capturing the imagination and attention of researchers as they probe the implications of quantum mechanics on these enigmatic cosmic entities.

2. Harmonic Oscillator

2.1. Harmonic Oscillator in Quantum Mechanics

Physicists may meet with a somewhat difficult problem, which will happen on the harmonic oscillator in a quantum world. Before delving into the quantum rendition of the system, it is regrettable that a limitation in extending prior methodologies appears. This stems from the fact that the kinetic equation at the disposal within the realm of quantum mechanics is confined to the Schrödinger equation, which states that

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi. \quad (1)$$

Recall the Hamiltonian in classical mechanics

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad (2)$$

it is easy to get the equation in quantum mechanics after operatizing quantities in Hamiltonian

$$i\hbar \frac{\partial}{\partial t} \psi = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \right) \psi. \quad (3)$$

Thus, solving the stationary Schrödinger equation is the only thing to do.

In fact, there are many methods to solve this differential equation, but this paper does not focus on solving a differential-equation problem, and instead consider the commutator $[x, p] = i\hbar$, where x is the position operator and p is the momentum operator. Introducing the annihilation and creation operators [1], it follows that

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right), a^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right). \quad (4)$$

Notice that the commutator $[a, a^+] = 1$, one thus introduces the particle number operator $N = a^+ a$. It is transparent that $[N, a] = -a$, $[N, a^+] = a^+$. Choose the eigenstate of N as the base state, it is easy to prove $Na\psi_n = aN\psi_n - a\psi_n = (n-1)a\psi_n$ and $Na^+\psi_n = a^+N\psi_n + a^+\psi_n = (n+1)a^+\psi_n$. There exists the relation below $a\psi_n = \sqrt{n}\psi_{n-1}$, $a^+\psi_n = \sqrt{n+1}\psi_{n+1}$, with the normalization factors. So, in the physics picture, a and a^+ really represent creating or annihilating a particle respectively. And then, one can realize that the Hamiltonian is decided by the particle number N :

$$H = \left(N + \frac{1}{2} \right) \hbar\omega \quad (5)$$

For now, one can get all ψ_n s, once the eigenstate ψ_0 is known, it is directly to find that $a\psi_0 = 0$, i.e., $\left(x + \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \psi_0 = 0$. This indicates

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar} x^2} \quad (6)$$

Then one can use operator a^+ to get ψ_n immediately. Fig. 1 presents the first a few wave functions for the harmonic oscillator.

2.2. Quantum Field Theory

To not digress, the paper only mentions some fundamental concepts in quantum field theory, especially focus on why QFT is a ‘oscillator-like’ thing. It is evident that the Schrödinger equation does not align with the principles of Special Relativity. This discrepancy arises primarily from its adoption of the classical mechanics form of the Hamiltonian. Upon adopting Special Relativity, the Hamiltonian must adhere to the following expression

$$H^2 = \vec{p}^2 + m^2. \quad (7)$$

So, the motion equation has the form

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)\psi = 0, \quad (8)$$

which is well known as Klein-Gordon equation. What should be stressed here is that the introduction of Klein-Gordon equation above somewhat too classical. In QFT, it is the translation invariance produce the Klein-Gordon equation for free field directly. Then the question is, what makes the QFT oscillator-like and the key is Fourier transformation.

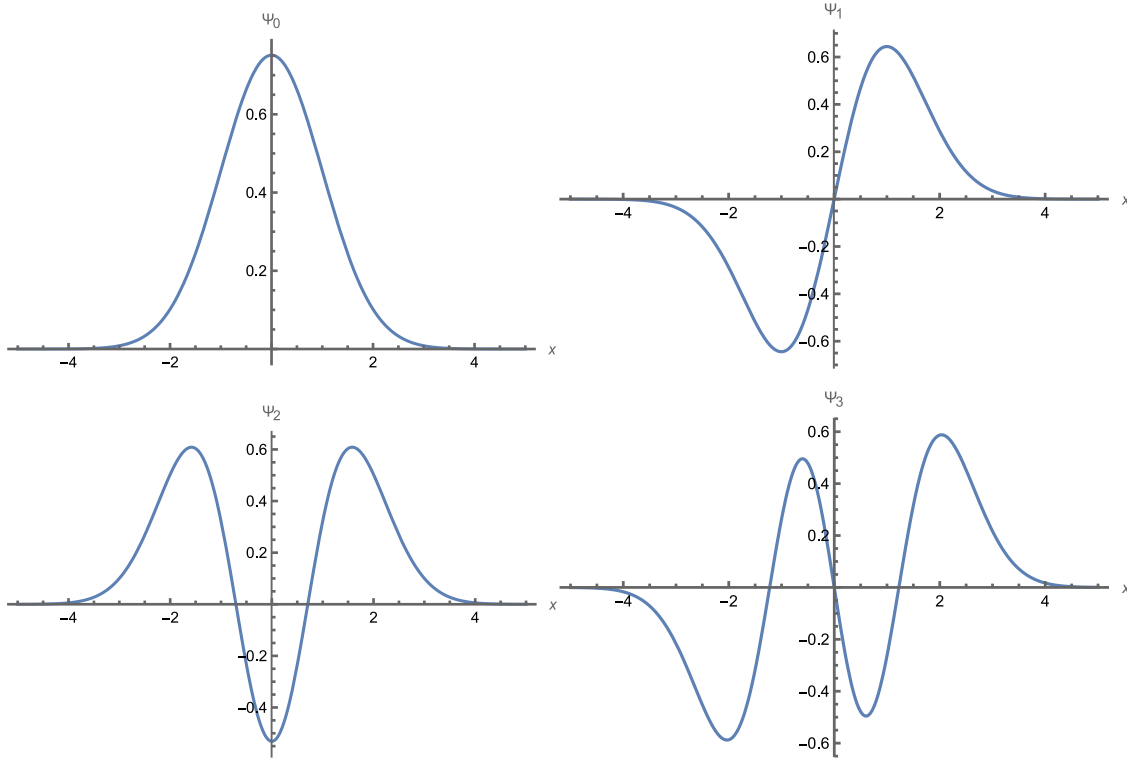


Fig. 1 Illustration of the wave functions with quantum number $n=0, 1, 2$, and 3 .

Consider the field

$$\psi(x, t) = \int d^3 p e^{ipx} \phi(\vec{p}, t). \quad (9)$$

One does not need to integral the energy p^0 , since the field is on-shell. Calculate the Klein-Gordon equation with the expression above, one can find that

$$\left(\frac{\partial^2}{\partial t^2} + \vec{p}^2 + m^2\right)\phi(\vec{p}, t) = 0. \quad (10)$$

One can identify that it is a harmonic oscillator immediately. Remember that basically the harmonic oscillator in quantum mechanics is depicted by the creation and annihilation operators, one can easily imagine that depicting $\phi(\vec{p}, t)$ in the same way is quite possible, although it is very

difficult in the history. Thanks to the practice of physicist seniors, one can suppose the field has the form [2]

$$\psi_a^+(x) = \sum_{\sigma,n} \int dp u_a(x,p,\sigma,n) a(p,\sigma,n), \psi_a^-(x) = \sum_{\sigma,n} \int dp u_a(x,p,\sigma,n) a^+(p,\sigma,n), \quad (11)$$

where the a and a^+ are annihilation and creation operators respectively. This basically argues for that the field is constructed by the annihilation and creation of particles.

The important thing about u_a and v_a is the transformation properties of the field. The transformations accord with Special Relativity construct the Lorentz group, and the most important part of Lorentz group is Poincare group. Supposing all fields should transform like

$$U(\Lambda, b) \psi_a^\pm(x) U^{-1}(\Lambda, b) = D_{ab}(\Lambda^{-1}) \psi_b^\pm(\Lambda x + b) \quad (12)$$

where U represents the representation of Poincare group in operator space. It is the same to all fields, and D_{ab} is the representation of Poincare group in vector space (R^n), which is different for different fields. For instance, to scalar field, D_{ab} is just the identity matrix, and to electromagnetic field, $D_{ab}(\Lambda)$ is Λ itself. While the situation is different for the field describes electrons, the representation becomes more complex and the representation is non-unitary, which is the most significant difference from the former two fields.

As an example, the translation transformation decides

$$u(x, p) = u(p) e^{ipx}, v(x, p) = v(p) e^{-ipx} \quad (13)$$

for any free fields. Here the labels n and σ have been drop. The discussion above is based on free fields, then how about the situation with interaction? Things goes easily, because in quantum mechanics, people learnt the interaction picture $\psi_I = e^{iH_0 t} \psi_S$, where I and S represents the state is in interaction and Schrödinger picture respectively. The interaction is so convenient since it separates the evolution of operator and state. The Schrödinger equation then becomes

$$i\hbar \frac{\partial}{\partial t} \psi = V \psi \quad (14)$$

in interaction picture, where V is the interaction in interaction picture. Thus, the Heisenberg equation becomes

$$i\hbar \frac{\partial}{\partial t} A = [A, H_0]. \quad (15)$$

So apparently, the operator obeys the same equation with free fields, so people can use the form of fields produced in free field theory fortunately. Then the Schrödinger equation produces the time evolution operator $u(+\infty, -\infty) = T[e^{-i \int dt V}]$. Usually, people use the interaction density $u(+\infty, -\infty) = T[e^{-i \int dx V}]$, where the x is the spacetime coordinate, and T is the time ordered product. From this one can calculate the time evolution operator from distant past to distant future (S -matrix). By adopting Wick theorem, one can find that S -matrix consists of many parts. And each part is expressed by an integral formula. The paper will not delve into the specifics of solving these integrals but will introduce a captivating concept known as Feynman rules. This innovative idea represents integral formulas through diagrams, wherein each component of the diagram possesses its distinctive form. Employing Feynman rules is akin to utilizing a multiplication table, where one simply matches the components and records the resulting outcome.

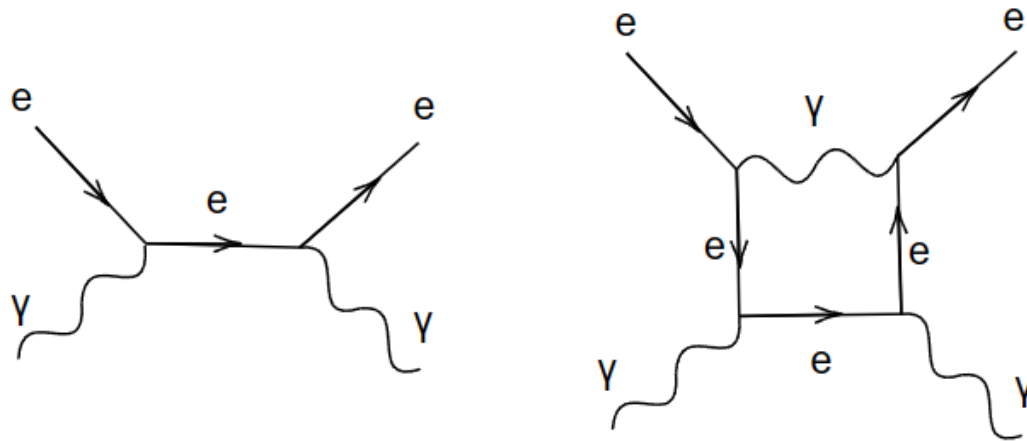


Fig. 2 Feynman diagram of Compton effect (up to the fourth order).

For example, Compton effect is no doubt a famous and important physics experiment, its Feynman diagrams are (up to the fourth order) shown in Fig. 2. It symbolizes the sequence in which an electron absorbs a photon and subsequently emits a photon, encapsulating the interaction between the electron and the electromagnetic field. Naturally, there exists a myriad of Feynman diagrams, with each diagram depicting a distinct and intriguing physical process. The left graph shows the second order Feynman diagram of Compton effect, whilst the right illustrates the fourth order one. At the last of this section, it is important to stress that the real field operator is not a individual ψ^+ or ψ^- , but the combination of these two operator. In QFT, it is to adapt to the causality in special relativity. And this indicates that there is no field only create or annihilate particles.

2.3. Expand to Gravity

It is widely acknowledged that the Standard Model predicts the existence of 62 types of fundamental particles, with experimental evidence confirming the existence of 61 of them. Remarkably, the graviton remains the sole elusive particle that has eluded detection thus far. It is somewhat ironic that the earliest-recognized interaction has now become one of the most enigmatic. Nevertheless, physicists continue their relentless pursuit of understanding gravity's essence, leading to numerous endeavors aimed at unraveling its mysteries.

String theory is one of the candidates for gravity theory, and optimists may deem it as a final theory of physics, although it lacks practice. Recently, Jonathan Oppenheim of University College London proposed a novel approach to elucidate gravity. In his paper [3], Oppenheim endeavors to reconcile general relativity and quantum mechanics by extending the Heisenberg equation of the density matrix to the master equation. This extension involves the introduction of additional terms into the Heisenberg equation while ensuring the preservation of probability. To keep curious is human, so exploring the rule of the world, the process, will continue forever.

Nowadays, there are many methods which help people to study gravity, such as gravitational wave detection. Nevertheless, people are interested in exploring the area by theoretic way. Recollecting the process people construct the statistic mechanics, people first realized the macro nature of a system, such as $PV = NkT$, before figuring the microscopic properties out. Hence, a natural approach emerges rather than directly speculating about the microscopic properties of gravity, it is prudent to first explore its macroscopic manifestations. In this regard, black holes stand out as the most crucial macroscopic systems that aid in the study of gravity. If physicists aim to detect the effects of gravity, the spacetime under consideration must either be sufficiently large or the energy involved must be sufficiently high. Conversely, if one seeks to detect quantum effects, a microscopic system or systems with extremely high energy levels are required. Therefore, in the quest to elucidate the essence of gravity, the requirement of a macroscopic system with high energy makes black holes the most compelling candidate for conjecturing the final theory of gravity. Since Hawking proved the first law

of black hole and associated the surface gravity κ to the temperature of the black hole in the meaning of black body radiation, black hole thermal dynamic became a hot research area soon [4, 5].

There are so many kinds of black holes are studied in the method of black hole thermal dynamic, such as Schwarzschild black hole, RN black hole, KN black hole, and the version with positive or negative cosmological constant Λ , and of course the version adopted the quantum effect.

3. Example for Quantized Gravity -- qBTZ Black Hole

It is well-known that the Einstein-Hilbert action is given by

$$S[g] = \int d^4x \sqrt{g} R. \quad (16)$$

One way to adopt the quantum effects is, add high curvature terms, like $R^2, R^{ab}R_{ab}$, in it. So, something goes interesting that sometimes the high curvature terms can be introduced from the projection from a high dimension spacetime. Just for example, the paper will introduce the quantum Banados-Teitelboim-Zanelli (qBTZ) black hole below, and before that the bulk spacetime will be mentioned, see Fig. 3.

The bulk spacetime is adopted, since the brane in the bulk cancels the infrared divergency in fact. By the way, it is very interesting that the infrared divergency corresponds to the ultraviolet divergency of the conformal field theory of the surface [6]. Therefore, the black hole the report focus on is the one on the brane, which is produced by the projection of the metric in AdS_4 (Anti-de Sitter) space time which satisfies the Einstein equation. And the metric in AdS_4 is [7]:

$$ds^2 = \frac{l^2}{(l+xr)^2} \left[-\frac{H(r)}{\Sigma(x,r)} (dt + ax^2 d\phi)^2 + \frac{\Sigma(x,r)}{H(r)} dr^2 + r^2 \Psi(x,r) \right] \quad (17)$$

where the functions are $H(r) = \frac{r^2}{l_3^2} + \kappa - \frac{\mu l}{r} + \frac{a^2}{r^2}$, $G(x) = 1 - \kappa x^2 - \mu x^3 + \frac{a^2}{l_3^2}$, $\Sigma(x,r) = 1 + \frac{a^2 x^2}{r^2}$, and $\Psi(x,r) = \frac{\Sigma(x,r)}{G(x)} dx^2 + \frac{G(x)}{\Sigma(x,r)} \left(d\phi - \frac{a}{r^2} dt \right)^2$.

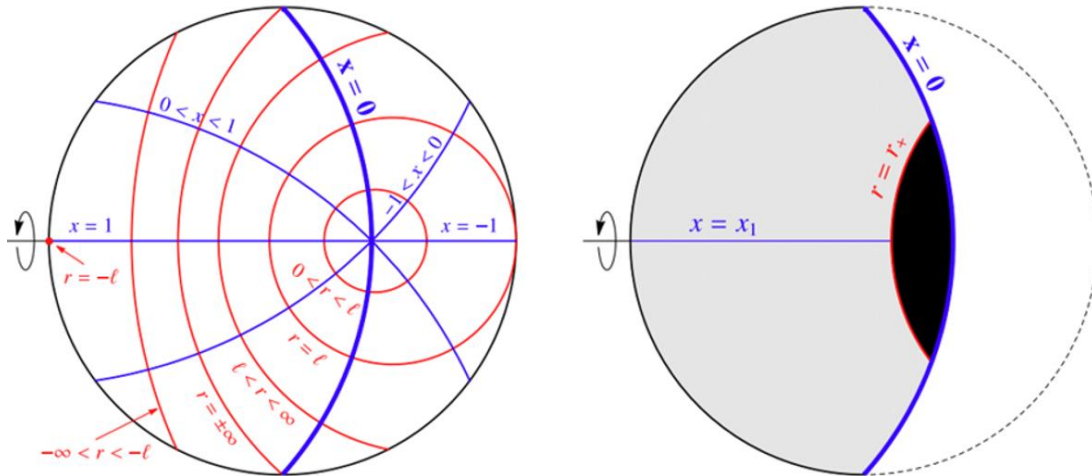


Fig. 3 Illustration of the Bulk spacetime [7].

However, when considering the induced metric on the brane at $x = 0$ [7]:

$$ds^2 = -\left(\frac{\bar{r}^2}{l_3^2} - 8g_3M - \frac{l\mu\Delta^2}{r}\right)dt^2 + \left(\bar{r}^2 + l_3^2 \frac{l\mu\tilde{a}^2\Delta^2}{r}\right)d\bar{\phi}^2 - 8g_3l\left(1 + \frac{l}{x_1r}\right)d\bar{t}d\bar{\phi} \\ + \left(\frac{\bar{r}^2}{l_3^2} - 8g_3M + \frac{(4g_3J)^2}{\bar{r}^2} - l\mu(1 - \tilde{a}^2)^2\Delta^4 \frac{r}{\bar{r}^2}\right)^{-1} d\bar{r}^2. \quad (18)$$

which satisfies [7]:

$$8\pi G_3 \langle T_{ab} \rangle = R_{ab} - \frac{1}{2}h_{ab}\left(R + \frac{2}{L_3^2}\right) + \\ + l^2 \left[4R_a{}^c R_{bc} - \frac{9}{4}RR_{ab} - \nabla^2 R_{ab} + \frac{1}{4}\nabla_a\nabla_b R + \frac{1}{2}h_{ab}\left(\frac{13}{8}R^2 - 3R_{cd}R^{cd} + \frac{1}{2}\nabla^2 R\right) \right]. \quad (19)$$

Thus, the projection produces the quantum effect, which is very interesting. Meanwhile, the quantum effects may make a difference on thermodynamics properties of black holes [8]. In their works, quantum effects cause a reentrant phase transition of qBTZ black hole in asymptotically Anti-de Sitter (AdS) spacetime, which is different from Hawking page transition of BTZ black hole without quantum effect. Moreover, the consideration of bulk spacetime is not solely motivated by quantum effects. As previously mentioned, the infrared divergence of an asymptotically AdS spacetime corresponds to the ultraviolet divergence of the conformal field theory (CFT) on its boundary, and this correspondence is not coincidental. Indeed, the AdS spacetime itself is conjectured to be dual to the CFT residing on its boundary [6]. Consequently, there is considerable interest among researchers in uncovering the thermodynamic duality between AdS spacetime and the CFT on its boundary, or more precisely, establishing a dictionary that relates the thermodynamic quantities of AdS black holes to those of the boundary CFT. Currently, numerous dictionaries have been proposed, yet none have garnered widespread recognition. The crux of constructing such dictionaries lies in establishing a variable that can connect the various quantities of a black hole to the central charge in the CFT. Presently, two approaches stand out as particularly convincing. One approach treats Newton's constant, G , as a variable [9], while the other introduces the conformal factor as an additional variable [10].

4. Conclusion

This paper primarily focuses on the concept of harmonic oscillators and their applications in theoretical physics. It highlights the extensive utility of harmonic oscillators and associated methodologies in various theoretical frameworks. Commencing with a concise overview of classical harmonic oscillators, the discussion transitions to their quantum mechanical counterpart. Subsequently, the paper delves into the application of harmonic oscillators in quantum field theory. Finally, it endeavors to extend the scope of harmonic oscillators to the domain of quantized gravity, illustrated with pertinent examples. In the final section of the paper, attention is drawn to the bulk spacetime and the intriguing duality between AdS spacetime and the boundary CFT. Specifically, the qBTZ black hole is presented as an illustrative example, derived from the induced metric on the brane $x = 0$, thereby incorporating quantum effects into the analysis. Subsequently, the AdS-CFT duality is elucidated, along with the two prominent approaches aimed at establishing dictionaries between AdS spacetime and the boundary CFT.

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