Research On Linear Potential Model and Its Application

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Abstract. This paper presents the results of research on the linear potential model. Firstly, the linear potential model is derived theoretically, and the unilateral and symmetric cases are discussed. The exact solutions, which are based on the Airy function, for both cases are obtained. By changing the mapping parameters, the author makes a comparative analysis between the curves of the wave functions, and the obtained results are consistent with the theoretical model. Next, the author analyzes the application of linear potential model in experimental control condition and talks about the application of simplification in theoretical model construction. To this end, four specific examples are selected to demonstrate the important role of linear potential. These include the study of neutron regulation in gravity field, the Stark effect formation in electric field, the one-dimensional Bose gas regulation in quantum chromodynamics, and the linear regulation of quantum dots and quark correlation model. Finally, the author also makes a prediction and prospect of the possible application of linear potential.

Keywords: Linear potential; Airy function; Energy level; Wave functions.

1. Introduction

In the field of quantum mechanics, the behavior of particles in the external potential energy field is often used as an important experimental observation object. However, in most cases, the behavior of particles is too complicated, and the wave function is too difficult to solve. For this reason, the linear potential model plays an important role in many cases. It provides an intuitive perspective for understanding how particles respond to simple potential fields [1]. This allows researchers to investigate the quantum behavior of particles under different potential energy conditions, such as bound states, scattered states, and quantum tunneling. The study of the analytical solution of linear potential model also contains the important idea of Wentzel-Kramers-Brillouin (WKB) approximation and provides ideological guidance for the construction of many complex theoretical models [2]. Based on this theoretical model and mathematical physics thought, the author will discuss its application in the real world and make reasonable predictions and prospects for its further application in the future.

The so-called linear potential means that the external potential field looks like $V(x) = ax + b$, in which its potential energy varies linearly with position [3]. Consider the one-dimensional steady state case, the Schrodinger equation can be reduced to the time-independent form. Throughout the paper, the discussion is divided into two cases based on the two potential energy scenarios.

This paper will start from the linear potential model itself, through the classification and discussion of the two cases. One will get the general form of the theoretical model, understand the ideological connotation of its modeling, and analyze the properties of the model with the help of images. Then, the paper will discuss the specific application of the linear potential model and summarize the application value of the model through four examples of experimental regulation and one application of theoretical modeling. Finally, the author will make predictions for development in the future.

2. Exact Solution of Linear Potential

2.1. The Linear Potential

The paper starts from the linear potential, which is given by $V(x) = qx$ for $x > 0$, and $\mathrm{V}(x)$ is zero otherwise. Specifically, $q = mg$ for the gravitational fields. Generally, there is zero probability when $x < 0$, so $\psi(0) = 0$. When $x > 0$, plug the potential energy expression into the original equation, one gets [4]

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + qx\psi = E\psi \tag{1}$$

Then the equation is treated without dimension. Let $x = sl_0$ and $E = \varepsilon E_0$, where $l_0 = (\frac{\hbar^2}{2qm})^{\frac{1}{3}}$ and the energy $E_0 = (\frac{q^2\hbar^2}{2m})^{\frac{1}{3}}$. Thus, the Eq (1) becomes $-\frac{d^2\psi}{dS^2} + s\psi = \varepsilon\psi$. Let $z = s - \varepsilon$, the equation finally become

$$\frac{d^2\psi}{dz^2} = z\psi. \tag{2}$$

The Eq (2) above is one of the simplest second-order linear differential equations. The British astronomer Airy first encountered it in optical research, the solution of the equation is therefore called Airy function, defined as [5]

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + zt\right) dt \tag{3}$$

and

$$Bi(z) = \frac{1}{\pi}\int_0^\infty [e^{\left(-\frac{t^2}{3}+zt\right)} + sin(\frac{t^3}{3} + zt)]\, dt\,. \tag{4}$$

Without considering the constraints, one can express the solution of the equation as a linear combination of the two Airy functions mentioned above. Namely, $\psi(z) = C_1 Ai(z) + C_2 Bi(z)$, where C_1 and C_2 are constants. However, if the physical meaning of a large case of z is considered as a constraint, the wave function corresponding to the solution represented by $Bi(z)$ in the Eq. (4) (assuming it is existent) cannot be normalized. Therefore, the author has eliminated the second half of the general solution to make the solution into a simpler form

$$\psi(z) = C_1 Ai(z). \tag{5}$$

The above part plays an important role in the WKB approximation. Researchers can linearize the potential energy at the transition point of the WKB approximation solution. Next, they solve the Airy equation and patch the wave function through its asymptotic expression to get the connection formula in the WKB approximation. However, the application of linear potential here is not only reflected in the same mathematical form, but also the contribution of its physical connotation is perhaps more intuitive and important.

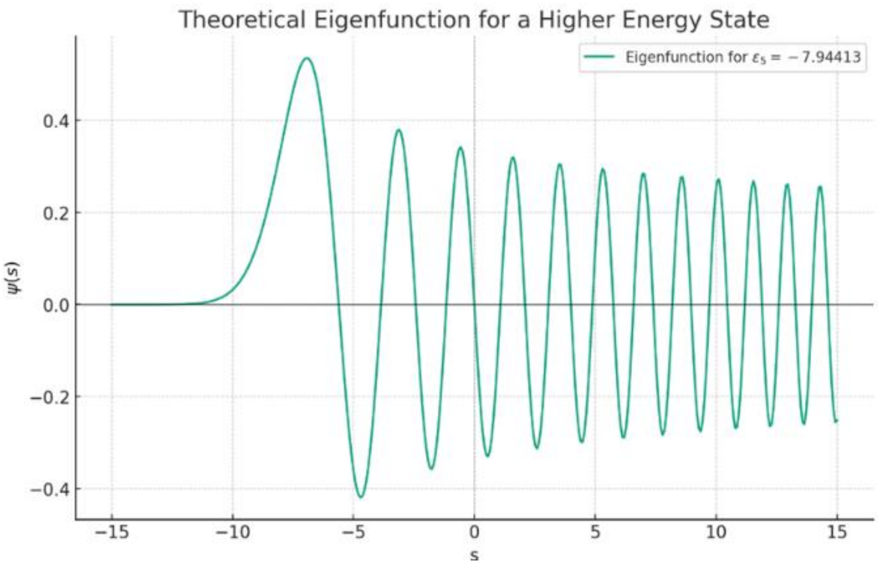

Fig. 1 The wave function for odd state.

2.2. The Symmetric Linear Potential

For a more general case with good symmetry, consider the symmetric linear potential of the form $V(x) = |qx|$ ($x \in [-\infty, +\infty]$). When $x > 0$, the same way can be used to solve this equation as before. Noteworthily, at this point where $x = 0$, the parity of the solutions before is worth of considering. Even and odd eigenstates for $x > 0$ can be extended to give the solutions for $x < 0$, via the relation $\psi^{even}(s) = \psi^{even}(-s)$ and $\psi^{odd}(s) = -\psi^{odd}(s)$.

With these properties, it then stands to reason that one needs the boundary conditions, which are

$$\psi^{odd}(0) = 0, \frac{d_{\psi^{even}}}{d_z}(0) = 0. \tag{6}$$

This equation can give the eigenvalues for odd and even parity states, respectively [6]. Using these two relations as constraints and querying the table of the zeros of the Airy function $Ai(z)$, the energy eigenvalues can be calculated, and the corresponding wavefunctions can be determined. As an example, the author chooses the fifth zero point ($\varepsilon_5 = -7.94413$) which corresponds to a higher energy eigenstate, see Fig. 1.

Following the above process, for an even state in a symmetric linear potential, the author selects the first classical zero $\varepsilon_1 = -1.01879$. The eigenfunctions for the odd and even states are shown in Fig. 2. This figure clearly differentiates the behavior of odd and even state wavefunctions in the potential. It not only highlights their symmetry and distinct characteristics around the origin, but also is in accordance with their respective boundary conditions. Then, the author changes the energy value of the even state and selects the fifth zero.

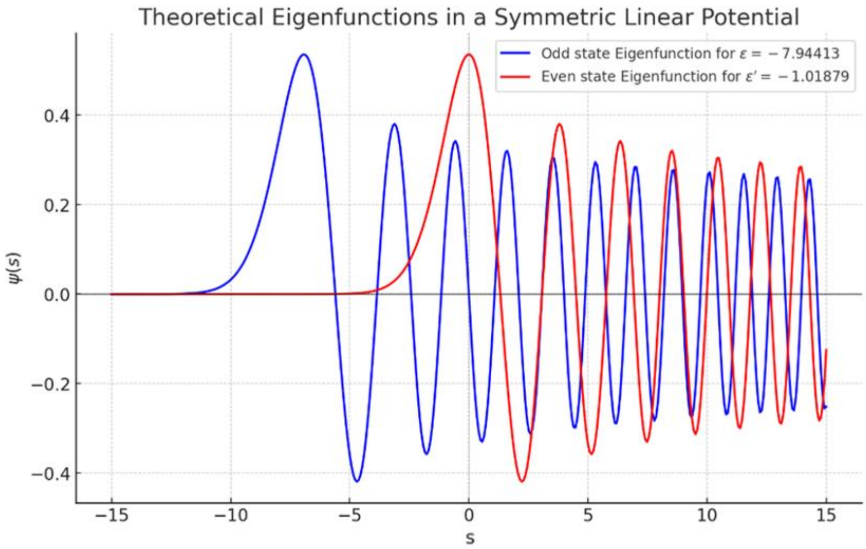

Fig. 2 The wave function for odd and even states.

The number of nodes (zero crossings) in the wavefunctions is a distinguishing feature of quantum states. In general, higher energy states have more nodes. The fifth even state is shown in Fig. 3. Interestingly, it shows more nodes compared to the specific odd state previously plotted. In addition, both wavefunctions exhibit decaying behavior away from the center, consistent with the confining nature of the linear potential. Moreover, this comparison between these three figures also highlights the rich structure of quantum states in a linear potential and illustrates how quantum mechanics describes particles with discrete energy levels.

3. Applications

Linear potential is widely used in theoretical and experimental physics, where it is commonly used for the approximate description of vibrational energy levels in molecular physics, the description of tunneling effects in semiconductor physics and nanotechnology, and the control of external potential fields in cold atom physics and quantum chromodynamics studies. The author will select several typical examples for analysis and discussion.

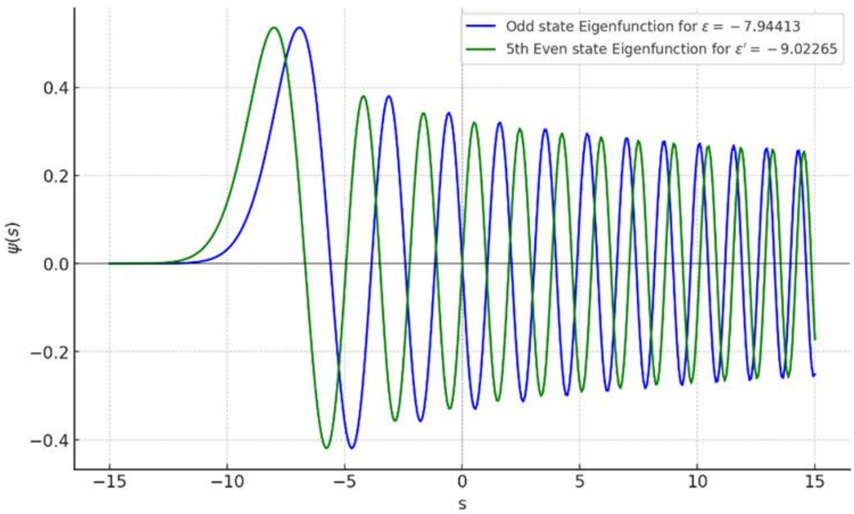

Fig. 3 The wave function for odd state and another even state.

3.1. Gravitational Field

In an experiment to verify the quantum-bound state of neutron gravity, the researchers combined a horizontal mirror with Earth's gravity to form a linear barrier [7]. This keeps the energy of the neutrons within a certain range (as shown in Fig. 4) and prevents them from escaping beyond the potential well. The researchers then analyzed in detail the quantum states of neutrons in the Earth's gravitational field, including their energy levels and wave functions, and explored the distribution of neutrons in different energy states and how these states are affected by the gravitational field.

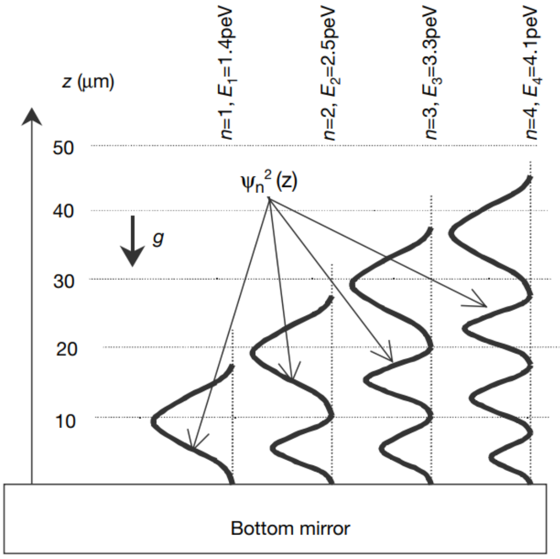

Fig. 4 Illustration of the experiment on measuring the particle in gravitational field [7].

Then, through experiments, the researchers obtained neutron counting rates at different heights (in micrometers), which are related to the height of the absorber. These experimental data agree well with previous theoretical predictions. Then, the researchers finally observed the 1.4 peV neutron ground state and hints of the first few excited quantum levels, confirming the existence of quantized energy levels for neutrons in a gravitational field. This result provides a new realm for verifying quantum mechanics. The linear potential is central to the study as it defines the gravitational potential well in which the neutrons are trapped. At the same time, it also provides an important help to explain the energy level splitting of neutrons in the gravitational field due to the principle of quantum superposition of states in quantum mechanics.

3.2. Electric Field

The researchers also simulate the external electric field with a linear potential model and explore the second-order shift of the energy spectrum due to external constant forces in a one-dimensional model quantum mechanical system [8]. This is known as the Stark effect. Although the author only carried out theoretical deduction and did not really use linear electric field to construct the potential barrier, the linear potential caused by external electric field is still an easy experimental constraint.

3.3. Quark Confinement in Quantum Chromodynamics

In a related cutting-edge study of a bose gas [9], the authors use a linear potential model to construct a potential box for a one-dimensional Bose-Josephson gas (which contains a tunable central barrier) to achieve a one-dimensional Bose-Josephson junction. The potential energy model consists of two Gaussian barriers, located at $x = 0$ and $x = L/2$, forming an effective box potential energy where the central barrier $V(t)$ is tunable. By changing the initial population imbalance (controlled by Voffset) and the barrier height, the authors investigate the kinetic behavior of one-dimensional bose gases, see Fig. 5. At large barriers, the two-mode model accurately captures the dynamics, while at low barriers, the dynamics involve dispersed shock waves and solitons. The adjustment of the linear potential allows researchers to explore different dynamical regions from weak to strong coupling, as well as Bose-Josephson oscillation and shock wave dynamics within these regions.

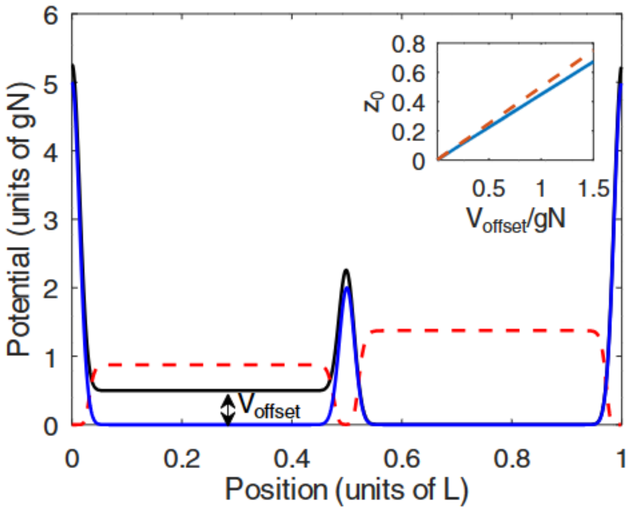

Fig. 5 The experimental setup for Bose-Josephson gas [9].

3.4. Control of Quantum Dots

The linear potential also makes a significant contribution to the realization of the lattice of quantum dots. In the study of electrical control of the uniformity of quantum dot devices, in order to overcome the problem that the spin qubits of quantum dots based on semiconductor technology are particularly sensitive to the local environment, the author obtains a highly uniform internal potential landscape electrically by the hysteretic change of gate voltage characteristics [10]. Using this method, the researchers homogenized the cut-off voltage in a linear array of four quantum dots, reducing the distribution range of the cut-off voltage by an order of magnitude. This practice, based on a linear potential model, allows researchers to precisely adjust and stabilize cutoff voltages in quantum dot arrays, which is critical for the scalability and performance of quantum computing devices. Further on this basis, the application of linear quantum dot arrays also has a broad space. In the study shown in Ref. [11], the author elaborated the simulation method of charge stability in a MOSfet linear quantum dot array. It also shows how to design and optimize quantum information processors through these methods. These studies above all demonstrate the fundamental role of simple linear potential models in complex application scenarios. It is believed that its innovative applications will continue to increase in the future.

3.5. Construct Theoretical Models

Although linear potential is a very simple theoretical model, thanks to its concise properties, people can use it to simplify the complexity of the model and make the results more intuitive when constructing theoretical models of many complex phenomena. For example, in a study of potential energy models for heavy quark elements such as the charmed quark and the bottom quark in high-temperature quark-gluon plasmas, the authors propose a heavy quark potential energy model for short and long distances [12]. The long-distance potential energy is modeled as a string that is shielded on the same scale as the Coulomb field. When constructing the heavy quark potential energy model, the authors consider the combination of linear potential (Coulomb potential) and nonlinear potential (string potential). This combination allows the authors to describe the kinetic behavior of heavy quark elements at different distance scales. In the study, the authors calculate the decay width of the quark even prime state, which is related to the imaginary part of the potential energy. Under high temperature conditions, it is the contribution of the linear potential that causes the imaginary part of the potential energy to increase, thus affecting the decay behavior of quark even elements.

4. Conclusion

By reviewing the above research and analysis of linear potential model and summarizing its application examples, it can be concluded that linear potential, as the most basic theoretical model, has an extremely wide range of applications. This is due to its characteristics of concise and efficient problem handling and its advantages of easy realization through technical means. Thus, it has achieved results in many fields of research and application. At the same time, the most basic analysis method of phenomena and the way of thinking of dealing with problems contained in linear potential model still contain huge potential value, waiting for further exploration and application in the future. According to the author's prediction, its future application scenarios include but are not limited to the study of the basic properties of condensed matter systems, and the simulation of qubit interactions. In addition, it is also helpful for the study of the properties of ultra-cold atomic systems such as Bose-Einstein condensation and Fermi condensation. Likewise, it is useful to the study of the transport behavior of photons in photonic crystals and other structures, as well as the exploration of topological material boundary states and topologically protected quantum states. Notably, in the era of continuous technological progress, linear potential model may become a basic idea that can solve problems across fields.

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