Exploring The Relationship Between Mathematics and Physics in The Special Relativity

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Abstract. The relationship between math and physics has always been a focus for scientists. This paper aims to deliver a general overview of the intersection of math and physics (in the age around the formation of special relativity) to inspire more people to have a deeper understanding of two subjects starting with the revolutionary birth and growth of 4-dimensional new physics in the 20th century. This paper discussed the relationship between mathematical models and physics by reviewing the evolution of special relativity and mathematical framework including $Minkowski$ spacetime and $\mathrm{Lorentz\ transformations}$. $\mathrm{Minkowski\ spacetime}$, the geometric framework of 4-dimension new physics, updated Lorentz transformation and generalized Einstein's special relativity. Furthermore, this paper reviewed changes in the relationship between math and physics and the relationship between math, theoretical physics, and experimental physics throughout time, providing insights from different scientists about the relationship between the development of physics and math nowadays.

Keywords: Special Relativity; Mathematics; Physics.

1. Introduction

Relativity, a theory that describes the relationship between time, space, matter, and energy, is the foundation of modern physics. 1905 is a big year for the physics field. Albert Einstein first published his paper that implied the basic structure of special relativity [1]. After Einstein's structure, math codifies the core structure of special relativity and generalizes it.

With advanced mathematical structures, modern physics has changed people's understanding of space, time, and reality from different perspectives. For example, people formed the idea of the mathematical concept of a spacetime continuum as the foundation of more general relativity, a mathematical perspective for the postulation of relativity, and further formalism of gravity in future studies [2, 3]. Special relativity is the start and milestone of the exploration of a fresh new structure of space and time.

In previous studies of the relationship between math and physics, scientists focused on Newtonian theories, general relativity, and theoretical physics theories like quantum field theories and string theories [4]. However, there is a lack of reflection on how human beings have evolved physics theories. This paper will explain and emphasize the importance of mathematical structure in the evolving process of special relativity. Therefore, this paper will provide an in-depth view of another period, the formation of the start of relativity theory. This paper will discuss the interplay between special relativity, Lorentz transformation, and $Minkowski\ spacetime$ [5].

This paper first lists basic knowledge of special relativity including Einstein's special, special relativity combined with Lorentz transformation and Lorentz transformation combined with $Minkowski\ spacetime$, achieving application of special relativity in four-dimension. Furthermore, this paper will discuss the historical background of the discovery of special relativity, and how exactly math models helped with special relativity's completion [5]. At last, this paper will provide shifts in the relationship between math and physics throughout history and how nowadays scientists view the shifts.

2. Basic Knowledge of Special Relativity

2.1. Postulation of Special Relativity and General Relativity

First, the speed of light is constant [1, 6].

The very first premise of everything in Einstein's relativity theory is that the speed of light in a vacuum is constant (c) regardless of the motion of the source or observer.

Second, physics laws are same in inertial reference frame. Mathematically,

$$\Phi'(x', y', z', t') = T[\Phi(x, y, z, t)] \tag{1}$$

Where $\Phi'(x', y', z', t')$ and $\Phi(x, y, z, t)$ are physics laws of two different inertial frames, and T is transformation function. T can only be Lorentz transformation in special relativity, the paper will discuss Lorentz transformation in chapter 2.

The formula above shows in the first inertial frame $\Phi(x, y, z, t)$ described by x, y, z, and t in the rule of Φ can be transformed to another inertial frame $\Phi'(x', y', z', t')$ using transformation function T.

The second postulation is considered as the covariance of physics laws. In other words, under the transformation of different reference frames, physics laws' forms are considered the same [6].

2.2. Einstein's Special Relativity

2.2.1 Simultaneity

Assume there are two different points A and B in the space, with respectively clock showing t_A and t_B. The paper also assumes the time that a ray of light starts at B and ends at A at the time of $t_A{}'$, and thus, the time interval of this ray of light which starts at B and ends at A is $t_A{}' - t_B$ [1].

The time interval of a ray of light which starts at A and ends at B is $t_B - t_A$.

Therefore, we define time at A and B as synchronized if:

$$t_B - t_A = t_A{}' - t_B \tag{2}$$

For a further explanation, Einstein used time interval to examine the concept of "Simultaneity" because time interval is the time that can associate time at A and time at B.

2.2.2 Time dilation

Einstein's paper On the Electrodynamics of Moving Bodies intuitively describes the concept: for a resting observer, the "clock" of a moving subject runs slower [1].

2.2.3 Length contraction

Einstein's paper On the Electrodynamics of Moving Bodies, intuitively describes length contraction: for a resting observer, the length of a moving object appears shorter.

Einstein didn't come up with the formula for length contraction in his early papers, but the formula was completed after Einstein integrated his work with Minkowski's mathematical framework, which this paper will discuss in 2.1 [1].

2.3. Lorentz Factor

Lorentz factor has existed for the description of the relation between Δt (time interval measured in rest frame) and $\Delta t'$ (time interval of inertial motion reference frame).

This paper will discuss the Lorentz factor's origin in 2.3.

$$\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} \tag{3}$$

Where: v = relative velocity between the object and observer; c = speed of light in a vacuum.

2.4. Time Dilation Formula with Lorentz Factor.

Mathematically, time dilation can be described as:

$$\Delta t' = \gamma\Delta t = \frac{\Delta t}{\sqrt{1-(\frac{v}{c})^2}} \tag{4}$$

When $\Delta x = 0$, Δt is time interval measured in the rest frame, and $\Delta t'$ is time interval of inertial motion reference frame [7]. This formula shows that the observed time interval $\Delta t'$ of the moving inertial frame will be longer than observed time interval Δt (also called proper time) measured in the rest frame. The dilation of time depends on the Lorentz factor.

2.5. Length Contraction Formula with Lorentz Factor

Mathematically, length contraction can be described as:

$$l = L/\gamma = L\sqrt{1-(\frac{v}{c})^2} \tag{5}$$

When $\Delta t = 0$, L is proper length, which is the length measured in the rest frame, and l is length in the inertial motion reference frame [7]. This formula shows that the observed length l of the moving inertial frame will be shorter than observed length L measured in the rest frame. The length contraction depends on the Lorentz factor.

2.6. Energy-Momentum Relation

3-momentum (momentum in three dimensions) can be written as:

$$\mathbf{P} = m\gamma\mathbf{v} \tag{6}$$

Where $\mathbf{P}$ is 3-momentum, **v** is velocity of an object, γ is the Lorentz factor, and m is mass of it [8].

Using Taylor expansion:

$$P^0 = \frac{mc}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{c}(mc^2 + \frac{1}{2}mc^2 + \cdots) \tag{7}$$

The second term within the Taylor expansion is the representation of non-relativistic kinetic energy, also because all components of 4-momentum P are conserved, implying P^0 is an energy particle [8]. So,

$$P = \begin{pmatrix} E/c \\ \mathbf{P} \end{pmatrix} \tag{8}$$

Combining three equations above:

$$E = m\gamma c^2 \tag{9}$$

When the object is at rest the formula can be written as:

$$E = mc^2 \tag{10}$$

Where E is energy, m is mass of the object, γ is the Lorentz factor and c is the speed of light [8].

3. High-dimensional Topological Mathematical Models and Their Applications

3.1. Minkowski Spacetime

Minkowski spacetime is the geometric framework of special relativity [5]. Minkowski defined a four-dimension vector (x, y, z, ct), where c is the speed of light and t is time.

The definition of the four-dimension vector implies the existence of four-dimension coordinates and metric tensor η.

Mathematically, flat Minkowski metric (η) can be written as:

$$\eta = diag(1, -1, -1, -1) \quad (11)$$

Compared to the three-dimensional description in derivatized form:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (12)$$

Where ds^2 is invariant interval in spatial dimensions, $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$, $dx^\mu dx^\nu$ are components of dx [8].

For examples, spacetime interval between two points $A(xA, yA, zA)$ and $B(xB, yB, zB)$ is:

$$c^2(t_A - t_B)^2(x_A - x_B)^2(y_A - y_B)^2(z_A - z_B)^2 \quad (13)$$

Where t_A and t_B are time at A and B.

If the separation of two events delta s^2 is bigger than zero, these two events are time like separated, which means they are closer in space rather than in time [8].

If the separation of two events delta s^2 is smaller than zero, these two events are spacelike separated, which means they are closer in time rather than in space [8].

If the separation of two events delta $s^2$is equal to zero, these two events are lightlike separated, which means they can be connected by light according to Minkowski spacetime [8].

3.2. Lorentz Transformation on one Axis

Assume an object is moving only on x axis with a constant speed v:

$$x' = (x - vt) / \sqrt{1 - (\frac{v}{c})^2}$$
$$y' = y$$
$$z' = z$$
$$t' = (t - vx/c^2) / \sqrt{1 - (\frac{v}{c})^2} \quad (14)$$

Where x, y, z, v, t are displacement, velocity, and time in one inertial frame, while x', y', z', v', t' are in another. c is the speed of light [9]. Formulas (14) above are Lorentz transformation for the object in spacetime.

Velocity transformation:

$$v_x' = (v_x - v) / (1 + vv_x/c\text{^}2)$$
$$v_y' = v_y / \sqrt{1 - (\frac{v}{c})^2}$$
$$v_z' = v_z / \sqrt{1 - (\frac{v}{c})^2} \quad (15)$$

Where v_x, v_y, v_z are components of v on x, y, and z axis in one inertial frame, while v'_x, v'_y, v'_z are in another inertial frame [9]. Formulas (15) above are Lorentz transformation for the object in velocity.

3.3. Lorentz Transformation in Minkowski Spacetime

A general Lorentz transformation in four-dimension can be described by Metrix Λ, which represents rotation between space and time: On X axis

$$\Lambda = \begin{vmatrix} \gamma & \frac{-\gamma V_x}{c} & 0 & 0 \\ \frac{-\gamma V_x}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{16}$$

Where γ is Lorentz factor,$v_x$is $x$component of velocity [8]. Substitute v_y and v_z into v_x to get Lorentz transformation matrix on y and z axis respectively. This matrix of Λ is an upgraded version of Lorentz transformation. A combination of special relativity and Minkowski spacetime can be conducted by combining Lorentz transformation and Minkowski spacetime. After upgrading Lorentz factor in a four-dimension vector, we can substitute this vector in spacetime and velocity formulas this paper have discussed in 2.2:

In a fixed frame S, general spacetime and velocity:

$$X(\tau) = \binom{ct(\tau)}{\mathbf{X}(\tau)}$$

$$v = \frac{d\boldsymbol{X}}{d\tau} = \binom{cdt/d\tau}{d\mathbf{X}/d\tau} = \frac{dt}{d\tau}\binom{c}{d\mathbf{X}/dt} = \gamma\binom{c}{d\mathbf{X}/dt} \tag{17}$$

Where τ is proper time, $X$is 4-spacetime, $\mathbf{X}$ is displacement, v is 4-velocity, γ is Lorentz factor, and t is time. Formulas above are description of 4-spacetime and 4-velocity.

In observer frame S':

$$v' = \Lambda v$$

$$x' = \Lambda x \tag{18}$$

Where v' and x' are observed velocity and spacetime, and Λ is Lorentz transformation in fourth dimension. Formulas (18) above are 4-dimension application of updated Lorentz transformation. Special relativity can be used in a more general way by integrating with Minkowski spacetime.

3.4. Minkowski Spacetime used in Energy-Momentum Relation

Three-dimension momentum is a three-dimensional vector, and four-momentum is a vector in four dimensions [10].

Three-momentum:

$$\mathbf{P} = (p_x, p_y, p_z) = m\gamma\mathbf{v} \tag{19}$$

Four-momentum:

$$P = \binom{E/c}{\mathbf{P}} = (E/c, p_x, p_y, p_z) \tag{20}$$

Use $Minkowski\ norm$ to calculate equation above:

$$P \cdot P = \eta_{\mu\nu}P^{\mu}P^{\nu} = \frac{-E^2}{c^2} + |\mathbf{P}|^2 = -m^2c^2 \tag{21}$$

Where: $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$, m is the mass of the object, c is the speed of light, $\mathbf{P}$ is 3-momentum, E is the energy, $P^{\mu} and\ P^{\nu}$are components of 4-vector P [8]. Formulas above show 4-dimension physics concepts that are in a new page in physics history.

4. Mathematical and Physical Relationships

4.1. Historical Background

Before1905, there are basic Lorentz transformations and Michelson-Morley experiment, which challenges the existence of luminiferous ether. This experiment inspired Einstein's thought experiments in the future [11].

In 1905, Einstein used 2 important postulations to conduct thought experiments and proposed the structure of special relativity by suggesting a paradox between classical physics and Maxwell's equations [1].

In 1908, Minkowski came up with a math frame for three-dimensional space, which allows the calculation of "distance" between spacetime [5]. The concept of spacetime continuum then generalized Einstein's relativity. Einstein published his paper about general relativity in 1915 [6].

After 1908, physicists combined the idea of 4-dimensional object that appeared in Minkowski spacetime as a framework and represented momentum in a 4-dimension way. For example, 4-momentum this paper has discussed in 2.4: formula (19).

4.2. Discussion of the Relationship Between Theoretical Physics and Math in Special Relativity

The relationship between basic Lorentz transformation and special relativity is codification: formula (4) and (5).

As shown the information above, Lorentz's transformation uses a mathematical method to codify Einstein's thought experiments about time dilation and length contraction in the paper On the Electrodynamics of Moving Bodies.

With Minkowski spacetime theory, Lorentz transformation was upgraded in four-dimension, being more generalized: formula (16) and (14).

Furthermore, the relationship between Minkowski spacetime and Einstein's special relativity could also be seen in relationship between $Cartisan$ coordinate system and Newton's laws, showing that math has applied physics in a more general level: formula (17) and (18).

Minkowski spacetime theory for Einstein's geometric frame, it will be very hard to apply Einstein's conclusions with a proper mathematical representation. Einstein's first paper about special relativity. On the Electrodynamics of Moving Bodies only consists of one formula for time dilation, but with Minkowski spacetime theory, Einstein's special relativity can be applied in four-dimension, helping Einstein come up with a more formal and generalized relativity: general relativity [1, 5].

In a more general perspective, Minkowski's spacetime is also the framework of fresh new physics in the 20th century by providing the idea of a 4-dimension representation integrating space and time. Momentum is written in a 4-dimension way combining space components and time components, filling the blank space of 4-dimension spacetime and velocity theories. For example, 4-momentum this paper has discussed in 2.4: formula (20).

Overall, in the age of special relativity, math plays the role of cornerstone not in the birth of new physics, but in the growth of new physics. Einstein came up with the basic descriptions of special relativity, and other scientists had the idea of 3-dimension momentum. With the incorporation of Minkowski spacetime as the geometric framework, 4-dimension new physics finally grew out of traditional 3-dimension physics with some vague ideas of the new physics.

4.3. Latest Updates of the Relationship Between Math and Theoretical Physics

Before the 1980s, math kept leading physics research because core mathematical results in physics research had already been discovered. Math served mainly as a tool to codify or predict in physics fields, and mathematicians could not learn anything significant new from physicists [4]. However, math turned out to be "inventions" in more advanced fields especially in physics. For example, complex numbers, linear operators, and algebras are defined by mathematicians and then widely used in physics [12].

After the 1980s, math and physics turned into a reciprocal relationship. Specifically, Knot theory, Moduli spaces of Riemann surfaces, and Morse theory in mathematics are used in physics, which was considered the “natural setting of math” at the time. Physics, on the other hand, provided a huge variety of generalizations for mathematical research. For example, research in relativity and electromagnetism inspired the discovered a new way of description of knot invariants; experimental facts and research of bosons and fermions in physics provided mathematicians insights into Morse inequalities [4].

The relationship between math and physics of being only reciprocal is no longer true after the proposal of string theory. After the proposal of string theory, physics seems leading the development of math by proposing “string quantum geometry” to describe the theoretical hidden spacetime that is integrated with quantum theory in string theory [4].

The trend that physics leads math can also be seen in scientists’ attitude toward future development of physics. Some scientists hold critical attitude toward developing physics based on mathematical structures.

For example, $Hossenfelder$ maintained a critical attitude about the fact that physics theories’ development was overly dependent on mathematical elegance [13]. The dependence will lead physics astray from scientific principles. Specifically, theories of supersymmetry, naturalness, and multiverse are not observed, but scientists choose to pursue these theories mainly because they have mathematical elegance. This will waste scientists time on more important topics [14].

4.4. Theoretical Physics, Experimental Physics, and Math

This paper has described how close theoretical physics and applied math are. Furthermore, people use experimental physics to bring theories closer the truth of reality [12]. Specifically, scientists used theories as predictive tool of nature by summarizing experimental facts and use experiments to confirm these predictions [4]. In Eugene Wigner’s opinion, human-beings will eventually approach “the ultimate truth” by constant exploration from theories to experiments. For example, Einstein was inspired by Michelson-Morley experiment which proved ether wrong and came up with relativity theories. Years after, perihelion precession of Mercury, gravitational redshift, and gravitational waves have proved general relativity right, and other experiments have proved equivalence principles in special relativity right.

In the process of interaction between theoretical physics and experimental physics, theoretical physics played role as applied math, generalizing the raw qualitative theory and providing predictions.

5. Conclusion

Considering the age of special relativity, math plays the role of cornerstone not in the birth of new physics, but in the growth of new physics. New physics in the age of special relativity was born due to the paradox between experimental evidence and old physics, and new physics were backward compatible to traditional physics. This paper discussed the interplay between mathematical models and physics: $Minkowski$ spacetime model generalized special relativity and inspired the birth of a more comprehensive 4-dimension with the help of Lorentz transformation while updating Lorentz transformation at the same time.

Furthermore, this paper reviewed changes in the relationship between math and physics. While math led the development of physics at first, the relationship shifted reversely in modern physics. Besides, this paper considers theoretical physics having the function of applied math and prediction in experimental physics.

This reviewed paper’s conclusion can be applied in research fields that focus on intersection of subjects between mathematical framework and physics research, guiding people to grasp a general view of interplay between math and physics to prepare for future studies.

In the future, physics will be leading the development of math, so mathematicians might consider reviewing more theoretical physics work to find the niche of researching area. In another perspective, the research of intersection between physics and technology might be more promising for two reasons: growth of artificial intelligence boosts its application in theoretical physics research, and development of physics is already leading math, suggesting that scientists should find new ways for future research.

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