Exploring The Foundations and Applications of Cosmological Simulation

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Abstract. This paper mainly discusses the theoretical basis of cosmology and the methods and techniques used in cosmological simulations. Based on Einstein's field equation and RW metric, Friedmann's equation is derived. People developed many universe models based on Friedmann's equation and came up with many possibilities for the evolution of the universe. With the development of computer science, now people can study cosmology using simulations. Different numerical techniques, such as the Monte Carlo method, Lagrangian method, etc., are used in the simulations. People can easily get access to data that is hard to gain from direct observations and understand the formation of all kinds of structures in the universe using simulations. In general, cosmological simulation has a great positive effect on the development of cosmology. This paper is for non-professionals and serves as a basic introduction to cosmology and cosmological simulation.

Keywords: Einstein's field equation, Friedmann's equation, universe models, numerical technique.

1. Introduction

Nowadays, cosmology is in a phase of rapid development. It only took 300 years from Newton's law of gravitation to the establishment of the big bang theory. Human's knowledge of the mysterious universe is constantly growing. However, there's still a lot human beings don't know about. One well-known mystery is that while 95% of the energy density in the universe, which is the so-called dark matter and dark energy, drives the accelerated expansion of the universe, people don't know what it is [1]. People can't even fully make sense of the formation of galaxies either. Furtherly, more and more observations, like the sharpened tension in inferred values of H_0, are found to suggest that the contemporary ΛCDM model is flawed [2]. To figure out all these, despite fitting the existing theories based on observation data, cosmological computer simulation is also needed, because it intuitively shows how the universe evolved and became the way it is now using numerical techniques. Numerical simulation techniques to study cosmology date back to 1975, when Peebles used 300 particles to simulate galaxy clusters, followed by the simulation of 1,500 particles within the universe [3]. With the maturity of technology, now cosmological simulation has gradually become an irreplaceable and efficient way to study cosmology. Cosmological simulation gives people a profound understanding of the formation of galaxies and shows their properties, which people may find hard to get from direct observations. For instance, using the approach of moving-mesh code, the simulation could produce the internal metallicity distributions within star-forming galaxies [4]. Besides, simulation is not just an extension of theories. By contrasting the differences between simulations in which parameters are deliberately adjusted, people can explore new physics and guiding opinions on the corrections of theories. Full physics simulation is composed of two parts (baryonic physics and dark-matter-only). By comparing full physics simulation and the dark-matter-only part, researchers can easily understand how baryon affects the universe [5]. To summarize, cosmological simulation has already become the bridge between theories and observations. Under the circumstance of the increasing computing power of supercomputers, cosmic simulation has a promising future and is believed to give answers to the key questions in cosmology.

This paper is written to review the historical development of cosmology and the positive role the simulations played in the development of cosmology. Thus, this research will focus on the basic theories of cosmology, and models and techniques that are used in simulations.

2. Basic Principles of Cosmology

Modern cosmology, also known as the Big Bang theory of cosmology, is based on Einstein's general relativity theory. Lemaître, who is a priest and cosmologist from Belgium, was the first to come up with the Big Bang theory. The name of this theory came from a broadcast from BBC, in which British cosmologist Hoyle strongly spoke against this theory, using the word "big bang" to show its absurdity [6]. However, both Friedmann's equation and the later observations proved Lemaître is right. Up to this day, the theoretical framework of this theory has been gradually developed. The paper will mainly introduce the basic principles of cosmology in the following paragraphs.

2.1. Cosmological Principle

The cosmological principle is two fundamental hypotheses: the universe is homogeneous and isotropic. The homogeneity means that there is no special point in the universe and every point in the space is the same. The isotropy means that the same universe can be seen from any direction. These two hypotheses are so important because all the theories of cosmology are based on them. Today, the isotropy of the universe has been testified by the cosmic microwave background (CMB) measurements on the magnitude of 10^{-5} [7]. Microwave detectors are used in telescopes to observe the radiation intensity and temperature of the CMB. It comes out that the same temperature is gobtainedfrom every direction, which is about $2.725K$, implying the isotropy of the universe. The cosmological principle is especially crucial when doing cosmological simulations, since it is the typical periodic boundary conditions, restraining the distribution of matter in the universe [1].

2.2. Friedmann's Equation

2.2.1 From Newtonian mechanics to Friedmann's equation

In 1686, Newton published his most famous work, "Mathematical Principles of Natural Philosophy". In this masterpiece, he proposed his metaphysical absolute view of time and space. Although in the perspective of general relativity, his philosophy is proven wrong, or to be more precise, narrow, he was still a pioneer and made a lot of progress in philosophy and physics at that time. This part intends to demonstrate how Friedmann's equation is derived from Newton's gravitation law, which may not be strict but still shows the consistency between Newtonian mechanics and modern physics.

According to gravitation law, the equation of motion can be expressed as

$$\frac{d^2l}{dt^2} = -\frac{Gm}{l^2} \tag{1}$$

where l is the distance between two mass points, G is the gravitational constant, and m is the mass of the central body. If both sides of this equation are multiplied by $\dot{l}$, then the equation looks like

$$\frac{d}{dt}\left(\frac{\dot{l}^2}{2}\right) = \frac{d}{dt}\left(\frac{Gm}{l}\right) \tag{2}$$

Integrate both sides, then

$$\dot{l}^2 = \frac{8\pi G}{3}\rho l^2 + C \tag{3}$$

where ρ is the density of the central body. According to Hubble's law, the universe is expanding, so the distance l is not constant. The scale factor $a(t)$ is introduced to describe the expansion. Thus, the distance can be expressed as

$$l(t) = a(t)d_c \tag{4}$$

Where d_c is constant and called comoving distance. Combining equation (3) and (4), the Friedmann's equation is derived

$$\dot{a}^2 + K = \frac{8\pi G}{3}\rho a^2 \tag{5}$$

Where $K = -C/{d_c}^2$.

The derivation above only uses gravitation law and Newton's second law. The more general and rigorous derivation of Friedmann's equation would use general relativity and Einstein's field equations.

In most simulations, the Newtonian system is preferable to the relativity system since it's a great approximation and relativity leads to loads of calculations. The linear structure growth speed is the same in both theories when matter is dominating, and the non-linear structure has a lower velocity which is much less than light velocity. Therefore, to some extent, Newtonian mechanics is still playing an essential role in cosmology.

2.2.2 Robertson-Walker (RW) metric

Thanks to the isotropy of the universe, the most general spacetime metric can be expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + sin^2\theta d\emptyset^2)] \tag{6}$$

Where k is the curvature of space. When $K = 0$, the space is flat, and the equation above degenerates into Euclidean space. When $K = 1$, this equation describes a closed universe like a spherical surface and when $K = -1$, it describes an open universe like a hyperbolic surface. The derivation of the RW metric does not depend on any physical assumption or additional condition but the cosmological principle. Therefore, as long as the universe is isotropic, the RW metric is applicative.

The geodesic equation of particles can be derived using RW metric

$$\frac{d^2x^i}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{7}$$

If the particle is still, then it will stay still forever, because when $\frac{dx^i}{ds} = 0$,

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{00}(\frac{dx^0}{ds})^2 = 0 \tag{8}$$

2.2.3 Friedmann's equation

Friedmann is a great cosmologist from Russia, who is the first to come up with a mathematical model of the universe. In 1922, inspired by RW metric and Einstein's field equation, he proposed his equation describing the possible circumstances of the universe's evolution. He successfully challenged Einstein's view of a static universe and introduced the term "expanding universe". His works were all the more valuable during those tough times.

Considering the nonzero component of Einstein tensor ($G_{\mu v} = R_{\mu v} - \frac{1}{2}g_{\mu v}R$):

$$G_{00} = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right] \qquad G_{ij} = -(\dot{a}^2 + K + 2a\dot{a})\bar{g}_{ij} \tag{9}$$

and the fluid form of energy and momentum:

$$T_{\mu v} = pg_{\mu v} + (\rho + g)U_\mu U_v \tag{10}$$

the Friedmann's equation can be derived from the time-time component of Einstein tensor, which is

$$(\frac{\dot{a}}{a})^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \tag{11}$$

According to the definition of Hubble constant ($H(t) \equiv \frac{\dot{a}}{a}$), the Friedmann's equation can be expressed as

$$H^2+\frac{K}{a^2}=\frac{8\pi G}{3}\rho \quad (12)$$

Thus, the age of the universe can be calculated the integral

$$t_0=\int_0^{a_0}\frac{da}{aH(a)} \quad (13)$$

2.3. Cosmological Models

In general, the cosmological model is a theoretical description of the motion and evolution of the universe on a huge scale. During the development of cosmology, many cosmological models have been put forward, and models have been gradually completed to fit the real observations. Before the introduction of all kinds of universe models, the author gives the derivation of some important parameters used a lot in the universe models.

Defining critical density $\rho_c\equiv\frac{3H^2}{8\pi G}$, density parameter $\Omega\equiv\frac{\rho}{\rho_0}=\frac{8\pi G\rho}{3H^2}$ and curvature parameter $\Omega_k\equiv\frac{-K}{a^2H^2}$, the Friedmann's equation can be transformed as

$$\Omega+\Omega_k=1 \quad (14)$$

By comparing Ω and 1 from the observation data, people will know the geometry of the universe. [7]. Combining equation (11) and the space-space component of the Einstein tensor, the acceleration motion equation is derived

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(\rho+3p) \quad (15)$$

Combing equation (15) and (11), the energy and matter conservation equation is derived

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0 \quad (16)$$

Only two of the three equations (equation (11), (15), (16)) are independent [7]. Thus, to build a universe model, an initial condition is needed. The following models mainly use the equation of state ($\omega=p/\rho$)as the initial condition.

2.3.1 The universe dominated by matter

If the energy density in the universe is mostly contributed by matter, then this era is called matter domination epoch. For matter, its state equation parameter $\omega=0$. Solving equation (11), (15), and (16), the evolution equation is derived

$$(\frac{\dot{a}}{a_0})^2={H_0}^2[1-2q_0+2q_0(\frac{a_0}{a})] \quad (17)$$

where $q_0=\frac{\Omega_{m_0}}{2}$. For more generalized cases, when ω is a constant, the equation above can be generalized as

$$(\frac{\dot{a}}{a_0})^2={H_0}^2[1-\Omega_{\omega_0}+\Omega_{\omega_0}(\frac{a_0}{a})^{1+3\omega} \quad (18)$$

The solution of this equation is

$$a(t)=a_0(\frac{t}{t_0})^{2/3(1+\omega)} \quad (19)$$

The famous Einstein-de Sitter universe is one of the matter-dominated universe models. To be specific, Einstein-de sitter universe is a special case of them when the universe is flat. Thus, when $k=0$, the Friedmann's equation degenerate as

$$(\frac{\dot{a}}{a_0})^2={H_0}^2(\frac{a_0}{a}) \quad (20)$$

And its solution is

$$a(t) = a_0(\frac{t}{t_0})^{2/3} \tag{21}$$

It shows that the universe has a starting singularity point and it'll keep expanding in the future.

2.3.2 The universe dominated by radiation

If the energy density in the universe is mostly contributed by matter, then this era is called the matter domination epoch. For matter, its state equation parameter $\omega = 1/3$. Solving equation (11), (15), and (16), the evolution equation is derived

$$(\frac{\dot{a}}{a_0})^2 = {H_0}^2[1 - q_0 + q_0(\frac{a_0}{a})^2] \tag{22}$$

Its solution is

$$a(t) = a_0\left(2H_0{q_0}^{\frac{1}{2}}t\right)^{\frac{1}{2}}(1 + \frac{1-q_0}{2q^{1/2}}H_0t)^{1/2} \tag{23}$$

2.3.3 The universe model containing the cosmological constant

When Einstein proposed his general relativity theory in 1916, It is commonly believed that the universe is still. However, to make sure the acceleration $\ddot{a} = 0$, Einstein had to put a cosmological constant in his field equation to provide the negative pressure. He corrected his field equation as

$$G_{\mu v} + \Lambda_{g_{\mu v}} = 8\pi g T_{\mu v} \tag{24}$$

Thus, the Friedmann's equation became

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{25}$$

Combining with the total state equation (including matter, radiation and cosmological constant), the equation could be written as

$$\frac{H^2}{{H_0}^2} = \Omega_{m_0}\left(\frac{a_0}{a}\right)^3 + \Omega_{r_0}\left(\frac{a_0}{a}\right)^4 + \Omega_{k_0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda_0} \tag{26}$$

where $\Omega_{\Lambda_0} = \frac{\Lambda}{3{H_0}^2}$ and $\Omega_{m_0} + \Omega_{r_0} + \Omega_{k_0} + \Omega_{\Lambda_0} = 1$.

It's worth mentioning that when the expansion of the universe is testified, Einstein deleted the cosmological constant and thought it was the biggest mistake in his life. However, later observations discovered the acceleration of the expansion, so the cosmological constant which provides the force in the direction of expansion accidentally accounts for the acceleration. Now, it is given a new name "dark energy".

2.3.4 ΛCDM model

ΛCDM model, also known as the cold dark matter model, is currently considered the most accurate model describing the evolution of the universe. This model is testified in many fields of cosmology and is the most used cosmological simulations' fundamental theory. This model includes radiation, matter, and dark matter and requires that $\Omega_m + \Omega_r + \Omega_\Lambda = 1$. If Ω_m and Ω_r are chosen to be the independent variables, then the universe evolution equation could be written as

$$\Omega_m' = \Omega_m(3\Omega_m + 4\Omega_r - 3) \tag{27}$$

$$\Omega_r'2 = \Omega_r(3\Omega_m + 4\Omega_r - 4) \tag{28}$$

where "$'$" is the derivative of the variable $N = \ln(a)$. Letting $\Omega_m' = \Omega_r' = 0$, three fixed points are got: $\Omega_{mc} = 0$, $\Omega_{rc} = 1$; $\Omega_{mc} = 1$, $\Omega_{rc} = 0$; $\Omega_{mc} = \Omega_{rc} = 0$, corresponding to the radiation domination, matter domination and cosmological constant domination respectively.

In this model, some assumptions are made about the dark matter. Firstly, it is non-baryonic.

It is composed of something other than protons and neutrons. Besides, it is cold. Its velocity is much lower than the velocity of light when the radiation equals matter. That's the reason why neutrino is not dark matter, although it's non-baryonic. What's more, it's nondissipative and collisionless, which means that it can't radiate photons, and dark matter particles interact with each other only through the weak force.

3. Cosmological Simulation Methods and Techniques

3.1. Initial Condition

Just like solving partial differential equations, cosmological simulations require boundary conditions and initial conditions. As mentioned earlier, the boundary conditions are given by the CMB. The initial position is given by

$$x = q + D(t)\Psi(q) \tag{29}$$

where x is the comoving initial position, q is the unperturbed particle position, $D(t)$ is the linear growth factor and $\Psi(q)$ is curl-free displacement field. The initial velocity is given by

$$a(t)\dot{x} = a(t)\frac{dD(t)}{dt}\Psi(q) = a(t)H(t)\frac{dlnD}{dlna}D(t)\Psi(q) \tag{30}$$

The form of the initial power spectrum can be derived from the inflation model as

$$P(k) = Ak^n|T(k)|^2 \tag{31}$$

where $T(k)$ is the transfer function and $n \approx 1$.

Two types of initial conditions are commonly used in simulations. One is uniformly sampled periodic large volumes initial conditions. It constructs a low-resolution background. The another is the zoom initial condition. Providing high resolution, it can give specific details about the regions of interest, such as dwarf galaxies [1]. Relying on the observation of the CMB and the perturbation theory, the temperature and density are assigned to dark matter and baryonic particles.

3.2. Numerical Techniques

3.2.1 N-body simulation

In many simulations like Illustris, there is a version of dark-matter-only simulation since dark matter is the main content in the universe. N-body simulation is often employed to track dark matter because this method can solve the collisionless Boltzmann equation [1].

Normally, the basic idea of the n-body simulation is to calculate the force field based on the distribution of particles and then calculate the integration of every particle to track its motion. However, the complexity of this algorithm is $O(N^2)$, which is too high to realize when dealing with a large number of particles [8].

Thus, particle mesh mothed is used to reduce the complexity. The basic idea of this method is to transfer the spatial distribution of the particles into density field on the mesh, based on which the force field is calculated. The complexity is reduced to $O(M^3 logM)$, where M is the number of cells in one direction.

- **Solving density field**

Let's say there are a bunch of particles whose density field can be expressed with δ function. Now, the space is divided with mesh, and each particle must belong to a cell. Then, the mass of each cell is calculated with the particles within it. Furthermore, to tell that the mass distribution of particles is different at different positions in each cell, a weighting operation is needed. A rectangle centered on the particle has many overlapping parts with neighboring cells. The mass of the particle is weighted by the area of the overlapping parts and spread to the neighboring cells (as shown in Figure 1). This process is similar to doing convolution, except that the kernel of the convolution is not a triangle, but a rectangle. Now, the density field, or in computer language, the array is calculated. In the array, each element represents the mass in a corresponding cel.

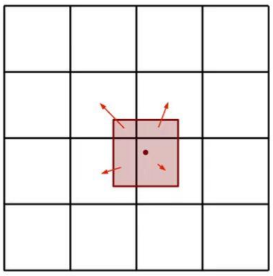

Fig. 1 Mass weighted distribution of particle mesh method.

- **Solving potential field and force**

With the density field, the way to solve the potential field is nothing but to solve the poison equation

$$\nabla^2\Phi(\vec{r}) = 4\pi G\rho(\vec{r}) \tag{32}$$

Under the Fourier transform, the derivation becomes four fundamental operations, so it's easy to figure out the image function. Then the primitive function can be derived by the inverse Fourier transform of the image function. And the force is the negative gradient of the potential field ($F = -\nabla\Phi$). It's worth noting that this force is the force that acts on a single cell. Since the mass of each cell is calculated using the weighting operation, dating back to the force on the particle still needs the weighting operation.

- **Time evolution**

After figuring out the force, the next step is to calculate the motion of the particles. Leap-frog method is commonly used here to calculate the integral, which can make the Hamiltonian constant during the integration.

Given a timestep Δt, the program will run using a loop algorithm. The algorithm within each loop can be divided into three steps. First, the particle moves with the current velocity within the first timestep

$$x_{i+0.5} = x_i + v_i\frac{\Delta t}{2}. \tag{33}$$

Second, according to the force of the particle, the velocity will evolve as

$$v_{i+1} = v_i - \frac{F\Delta t}{m}. \tag{34}$$

Finally, the particle moves within the second half timestep

$$x_{i+1} = x_{i+0.5} + v_{i+1}\frac{\Delta t}{2}. \tag{35}$$

After this algorithm is cycled for every particle, the N-body simulation is done.

3.2.2 Monte Carlo method

The Monte Carlo method is an estimation method using random sampling statistics. The basic theory of the Monte Carlo method is to carry out statistical analysis on a large number of random number samples, so the core idea of the Monte Carlo method is a random number. Only the random number in the sample has randomness, the obtained variable value can be credible and scientific.

Here is the basic principle of the Monte Carlo method. First, an appropriate probability model is established, which means to determine a random event A or random variable $X,$ so that the solution to be solved is equal to the probability of the occurrence of the random event or the mathematical expected value of the random variable. Then A simulation experiment is performed, that is, a random event A or a random variable X is simulated several times. Finally, the random experiment results are statistically averaged, and the frequency of A or the average of X are obtained as the approximate solution of the problem.

- **The problem with Lagrangian method**

Lagrangian method is a kind of numerical techniques. It is more flexible compared to the mesh method mentioned before because it allows the cells to move and deform. However, this flexible form generates a problem: The commonly used 'velocity field tracers' are found to not accurately follow the mass flow, since there are mass exchanges between cells due to the moving mesh. Thus, the Monte Carlo tracer is used to track those particles in this new condition.

- **Monte Carlo tracer**

Monte Carlo tracers work by being attached to specific resolution elements and exchanged between neighboring cells based on mass fluxes, allowing them to follow the fluid mass flow accurately. These tracers do not have their own phase-space coordinates but belong to fluid or other mass elements, ensuring that their spatial distribution aligns with the mass distribution [9].

The basic idea of the Monte Carlo tracers is that: When mass exchange occurs between cells, to accurately track fluid flow, tracking particles must be related to the proportion of mass exchange that occurs. For example, if there is a tracking particle α in the current cell i, and there is a mass exchange between cell i and cell j : if the mass flowing out of cell i is ΔM_{ij}, the probability of each tracking particle leaving cell i to reach cell j is the ratio of the mass flowing out to the current mass: $P_{ij} = \frac{\Delta M_{ij}}{M_j}$.

At the beginning, N tracer particles are assigned to each cell. Then, to decide whether the tracer α should leave cell i, we draw a random number $x_\alpha \in U(0,1)$. The tracer is moved to cell j, if the $x_\alpha < {P_{ij}}^{flux}$. Finally, we update the value of the reduced mass $\widetilde{M}_i$, to be $\widetilde{M}_i - \Delta M_{ij}$. The tracer is realized by looping through this algorithm.

3.2.3 Hydrodynamical method

The hydrodynamical method is often used to simulate baryons, which don't contribute much to the components of the universe but are visible and have a great effect on the physics in the universe. If the cosmic gas is considered as a continuum, then one can get a set of fluid equations, which contains an adiabatic state equation and equations of conservation of mass, of momentum, and energy. The hydrodynamical method is used to solve the set of equations, using the approach of discrete mesh [10]. Normally, the hydrodynamical method used in cosmology can be classified into three categories: Lagrangian method, Eulerian method and arbitrary Lagrangian-Eulerian method. The great difference between them is whether the mesh is free or fixed. Eulerian method considers the mesh fixed in the space and lets the particles pass through the mesh, leading to a mesh-based algorithm. On the flip side, Lagrangian method considers that the mesh is free, and the mesh can move with the particles, leading to a mesh-free algorithm [1]. The arbitrary Lagrangian-Eulerian method is like a combination of the former two methods. In some special regions like the surface, the mesh can move with matter, and in other directions, it can be fixed.

Under the different descriptions of the fluid motion, different forms of formulation are derived. In the Eulerian description, one can get the equation of conservation of mass

$$\frac{d\rho}{dt} = -\nabla \cdot (\vec{v}\rho) \tag{36}$$

of momentum

$$\frac{d\rho v}{dt} = -\nabla \cdot (\rho\vec{v}\otimes\vec{v} + P) \tag{37}$$

of energy

$$\frac{d\rho e}{dt} = -\nabla \cdot (\rho e + P)\vec{v} \tag{38}$$

where $e = u + v^2/2$ the total energy per unit mass.

In the Lagrangian description, one can get the equation of conservation of mass

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \vec{v} \tag{39}$$

of momentum

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\nabla P \tag{40}$$

of energy

$$\frac{De}{Dt} = -\frac{1}{\rho}\nabla \cdot P\vec{v} \tag{41}$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$.

- **Eulerian method**

Eulerian method is a traditional way to solve hydrodynamical equations, containing finite volume and finite difference method. For a conservation equation:

$$\frac{\partial f}{\partial t} = -\frac{\partial F}{\partial x} \tag{42}$$

one can do the Taylor expansion

$$f(x, t + dt) = f(x, t) - \frac{\partial F}{\partial x}dt + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\frac{\partial F}{\partial f}\right)dt^2 + O(dt^3) \tag{43}$$

The basic idea is that this equation is solved discretely on each cell to the second order accuracy. If the scale is small enough, then the solution can be considered smooth [11].

The advantage of this method is that the mesh is fixed, so there are no problems like mesh distortion or generating negative value of the area. However, this method allows the transportation of mass, momentum, and energy between cells. Thus, tracers like the Monte Carlo tracer mentioned before are needed to track the particles. What's more, it's harder to deal with the boundary conditions compared with Lagrangian method, because this method can't tell the position of the surface.

- **Lagrangian method**

In the Lagrangian method, the computational mesh is free to move with the matter and deform with it. This method is relatively computationally small, because there is no convective term, as shown in the equation (39)(40)(41). And since the mesh line always coincides with the surface of an object, the Lagrangian method is easy to deal with the boundary conditions. However, since the mesh is free, there are also problems like mesh distortion and mesh-tangling effects.

Smoothed particle hydrodynamics (SPH) is a widely used Lagrangian method. This method uses discrete particles to describe the macroscopic continuous distribution of the particles of the fluid.

In the SPH method, the following integration is used to show the value of function A at position $\overrightarrow{r_a}$:

$$A(\overrightarrow{r_a}) = \int_\Omega A(\vec{r})W(\overrightarrow{r_a} - \vec{r}, h)d\sigma \tag{44}$$

where W is the kernel of the integration and h is the smooth length.

The volume of each particle is $\frac{m_b}{\rho_b}$, so the equation (44) can be expressed using discretion form

$$A_a = \sum_b A_b W_{ab}\frac{m_b}{\rho_b} \tag{45}$$

The first derivative of the function A is represented by the following formulation

$$\frac{\partial A_a}{\partial x} = \sum_b \frac{m_b}{\rho_b}A_b\frac{\partial W_{ab}}{\partial x} \tag{46}$$

Equation (45) and (46) are two basic equations of the SPH method. By finding the numerical solution of these two equations, one can solve specific physical problems.

- **Arbitrary Lagrangian-Eulerian (ALE) method**

In this method, the mesh moves in any form in space [1]. It can move independently of the matter coordinate or the space coordinate. In this way, the moving interface of the object can be accurately described, and the reasonable shape of the element can be maintained by specifying the appropriate mesh motion form. Its nature is to add a diffusion equilibrium term to the original differential equation. The pure Lagrange method and the Euler method are two special cases of the ALE method, that is, the Lagrange method degenerates when the velocity of the mesh points is equal to the velocity of the material points, and the Euler method degenerates when the mesh is fixed in space.

4. Significance of Cosmological Simulation

4.1. Expansion of Observation Capability

To better study the physical, chemical, and geological properties of celestial bodies, it is particularly important to observe the data of celestial bodies. Although we can observe celestial bodies through astronomical telescopes, satellites, and other means, the number and information of celestial bodies that can be observed is still very limited. At this time, computer simulation technology has opened up endless possibilities for astronomical research, because people can get access to all the time domains in the simulations, which is impossible for direct observations. From simulations, people can take a look at the snapshots of the highly realistic universe and speculate the properties and characteristics of structures in the universe.

4.2. Study Deeper into Anomalies

Although the ΛCDM model has been testified in many fields in cosmology, there are still unexpecting phenomena. For example, the latest observations from JWST rejected the unrecognized crowding of cepheid photometry as an explanation for the Hubble Tension, implying that the current standard universe model is immature [11]. This problem is expected to be explained by cosmological simulations.

Continuing with an example, in the closest extrasolar system to Earth, the Hubble Telescope has found some extrasolar objects between Neptune and the asteroid belt, which are called KBOs. Exactly what effect these KBOs have, and how to explain it, remains a mystery. Through computer simulation, we can also simulate this phenomenon in a real sense, simulating star formation, star explosion, KBO movement, and so on.

4.3. Bring New Physics

Not like reality, people can change different parameters or components in the simulations. That brings people more possibility to get a profound understanding of the physical process that is happening in reality. For instance, as mentioned before, most simulations have a dark matter-only part, which neglects the effect of baryons. By comparing dark matter-only and full physics simulations people can directly understand how the baryons affect the evolution of the universe and the further essence of baryon physics. That may bring us new theories or corrections on the current theories.

5. Conclusions

In conclusion, modern cosmology is a study of the evolution of the universe based on differential geometry, linear algebra, general relativity, hydrodynamics, classical mechanics, and other disciplines. It provides a theoretical guide to human beings' understanding of the history of the universe. The cosmological simulation is essential for the development of modern cosmology. The state-of-the-art simulation repeatedly mentioned in this paper, like Illustris, really gives people ample data results and helps people testify or correct their theory, using a bunch of efficient numerical techniques. The simulations are mostly used in mimicking the formation of galaxies, black holes, and clusters, but the codes and algorithms invented for cosmology could be even used in hydrodynamics or high-energy particle studies.

To fully figure out the evolution of the universe, more and more simulations are being constructed aiming to simulate different epochs during the history of the universe. With them, people can make full use of the observation data from the James Webb Space Telescope and Hubble Space Telescope. The simulations are also expected to give answers to the key questions in cosmology such as the Hubble tension. Finally, the cosmological simulation has a promising future and there are still a lot of difficulties to be resolved in the next decade.

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