# Application Of Gauss's Law and Biot-Savart Law to Charged Bodies with Inhomogeneous Charges

Weiqian Peng *

Department of physics, Sichuan University, Sichuan, China

* Corresponding Author Email: 201141220096stu.scu.edu.cn

**Abstract.** This paper systematically discusses the application and importance of Gauss's law and Bio-Savart's law in electromagnetism. Firstly, the basic principles and mathematical expressions of Gauss's law and Bio-Savart's law are introduced, and their applications in simple cases are also introduced. Secondly, it describes the application in complex cases, such as how to find the electric field strength of an object with an uneven charge, in which calculus is used to calculate the charge from the charge density. Finally, it emphasizes the role of the two laws in promoting scientific research and technological innovation, and calls for further exploration of their application and significance to promote the development and innovation in the field of electromagnetism. This paper aims to give readers a comprehensive understanding and inspiration of Gauss's law and Biot-Savart's Law, explain the application of the two theorems, and provide useful reference and guidance for the research and practice in related fields.

**Keywords:** Electromagnetism; Gauss's law; Biot-Savart law; Calculus.

## 1. Introduction

The Gauss's law (also known as Gauss's theorem) is an important fundamental principle that describes the distribution and properties of electric or magnetic fields in space. The Biot-Savart law (also known as Ampere's theorem) is one of the fundamental principles that describe the behavior of magnetic fields caused by electric currents. These two principles play an important role in the interaction of electricity and magnetism. Through the in-depth understanding and application of Gauss's law, it is better to understand the nature of electromagnetic phenomena and lay a foundation for further research and application in related fields. The main application of the large charge efficient theory is to calculate the observed values of finite volume, large ground state energy with finite total charge [1]. By applying the insights provided by the Bio-Savart law, researchers and practitioners can reveal the complexity of magnetic field phenomena and pave the way for advances in a variety of technological fields.

In this paper, the application of Gauss's law to the calculation of electric fields generated by symmetrical charged bodies are presented. The application of Biot-Savart's law to the calculation and analysis of magnetic fields caused by current are also discussed. Therefore, the purpose of this paper is to comprehensively discuss the role of Gauss's law and Biot-Savart's law in electromagnetism, emphasizing their theoretical foundation, practical application, and contribution to the progress of science and engineering. Through this analysis, the article hopes to inspire further research and innovation in electromagnetism and related subject areas.

## 2. Theory of Gauss's Law

### 2.1. Gauss's Law

Gauss's law is based on Coulomb's law and superposition of field strength. Starting with the case of a special point charge at the center of a closed sphere (Gaussian surface), it is found that the electric flux (the number of electric field lines) passing through the sphere is equal to the algebraic sum of the charges contained in the sphere divided by $\varepsilon_0$. By using the superposition principle of field strength, former scholar can extend the case of point charge to point charge systems and even continuous charged bodies, and special spherical surfaces to general closed surfaces. The conclusion can also be described as the electric flux through any closed surface in vacuum is equal to the algebraic sum of the charges contained in the closed surface divided by $\varepsilon_0$. This statement is Gauss's law in vacuum, and its mathematical expression is

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0} \sum_S q_i \,. \tag{1}$$

When there is a dielectric, the dielectric will be polarized under the action of an external electric field, and there will be polarization charges $q'$ and depolarization fields $E'$, resulting in the formula (1) can not be used. This means that the $\sum_S q_i$ on the right side of the equal sign of the formula (1) cannot represent free charges. The polarization intensity vector $P$ must be introduced. The electric polarization intensity $P$ has an important application in the magnetic control of ferroelectric polarization [2], where $P$ represents the degree of polarization of the dielectric, $\oiint_S P \cdot dS = -\sum_S q'$. Thus, the electric displacement vector is defined as $D = \varepsilon_0 E + P$, where D plays a crucial role in finite-field calculations. It also has an application in the study of the electromagnetic properties of insulators [3]. Finally, the following formula is obtained by simply pushing to it, which can be expressed as Gauss's law in media

$$\oiint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \sum_S q_0 \,, \tag{2}$$

where $q_0$ is the free charge. Gauss law gives a method to calculate the field strength. In Gauss law, the field strength $E$ is an integrand function. If $E$ and $\cos\theta$ of any point on the Gaussian surface are constants, or the Gaussian surface can be divided into several finite regions and the surfaces of each region agree with $E$ and $\cos\theta$ are constant or equal to zero, $E$ can be extracted from the integral sign. After such mathematical treatment, the author can figure out the magnitude of $E$ on this surface. This paper will use Gauss's law to find some more complex electric fields.

### 2.2. Electric field of a Sphere of Uniform Charge Density

Since a sphere can be regarded as a superposition of an infinite number of tiny spherical shells, the article first considers the calculation of the field strength of a uniform charged spherical shell. The paper uses Gauss law to solve, and considering the symmetry of the sphere, it can be considered that the field strength on the sphere with the same distance from the center of the circle is the same, it is logically that $E_r$ can be defined as the strength of the electric field at a distance $r$ from the center of the circle, $R$ as the radius of the circle, and $\eta_e$ is the charge surface density. So according to Gauss's law $\oiint_S E \cdot dS = \frac{1}{\varepsilon_0} \sum_S q_i$, the author divides the discussion into two categories, which are $r < R$ and $r > R$.

When $r < R$, it is obvious that the charge is only on the surface of the shell, in Eq. (1), the right side of the equation is zero. Because of symmetry, the left-hand side of the equation becomes $E_r \cdot 4\pi r^2$. So $E_r = 0$. When $r > R$. The right side of equation 1 becomes: $\frac{\eta_e}{\varepsilon_0} \cdot 4\pi r^2$, so $E_r = \frac{\eta_e}{\varepsilon_0} \cdot \frac{R^2}{r^2}$.

As for the electric field of a sphere of uniform charge density, the author defines the $\rho_e$ as the electric charge density. It can also be divided into two categories: $r < R$ and $r > R$..

When $r < R$, a sphere is composed of an infinite number of spherical shells, and likewise the strength of the electric field generated by a sphere can be the electric field generated by an infinite number of spherical shells. So, in this situation, only the shell with a distance less than $r$ from the center of the sphere influences the electric field strength. By using Gauss's law for the part of sphere $(r < R)$, $E_r \cdot 4\pi r^2 = \frac{\rho_e}{\varepsilon_0} \cdot \frac{4}{3} \pi r^3$ and $E_r = \frac{\rho_e}{3\varepsilon_0} \cdot r$. It follows that in this case the electric field strength is proportional to $r$. The other case is $r > R$. Because of the symmetry, Gauss's law is used directly: $E_r = \frac{\rho_e \cdot R^3}{3\varepsilon_0} \cdot \frac{1}{r^2}$.

# 3. Applications of Three Different Laws

## 3.1. Biot-Savart Law

Biot-Savart law is the basic law of magnetic field generated by current-carrying wires. From 1819 to 1820, Danish scientist Osert published his own research results, which is the history of the famous Oster's law. Oster's experiments showed that electric currents can change magnetic fields, and scientists have since tried to quantify the relationship, and found the following formula

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\oint \frac{I\boldsymbol{dl} \times \boldsymbol{e_r}}{r^2}. \tag{3}$$

Biot-savart law and the superposition principle of magnetic field can be used to calculate the magnetic induction intensity of the magnetic field generated by any current carrying wire [4].

The Biot-Savart law can be used to find the magnetic field of the current carrying straight wire (wire thickness is ignored). Supposing a wire is infinitely long, and a constant current $I$ is flowing through it, the vertical distance between this point and the wire is $a$. Combine picture with vector algorithms, the direction of the elementary magnetic field $d\boldsymbol{B}$ generated by any current element $Id\boldsymbol{l}$ is the same, so converting the formula to scalar form

$$B = \frac{\mu_0}{4\pi}\int \frac{Idl\sin\theta}{r^2}. \tag{4}$$

From the Fig. 1, it is found that there are some equivalent substitutions, like $l = -r\cos\theta$, $a = r\sin\theta$. From these two equations, r can be replaced: $l = -a\cot\theta$. Take the differential form, it is directly to find that $dl = \frac{ad\theta}{\sin^2\theta}$. By replacing the above integral variable $l$ with $\theta$, a new formula is found [5]

$$B = \frac{\mu_0}{4\pi}\int_{\theta_1}^{\theta_2} \frac{I\sin\theta}{a} = \frac{\mu_0 I}{4\pi a}(\cos\theta_1 - \cos\theta_2), \tag{5}$$

where $\theta_1$ and $\theta_2$ are the values of $\theta$ angles at both ends of A and B. If the wire is infinitely long, $\theta_1 = 0$, $\theta_2 = \pi$, bringing into the calculation

$$B = \frac{\mu_0 I}{2\pi a}. \tag{6}$$

This result shows that the magnetic induction intensity $B$ around a current carrying infinitely long straight wire is inversely proportional to the first square of the distance $a$.

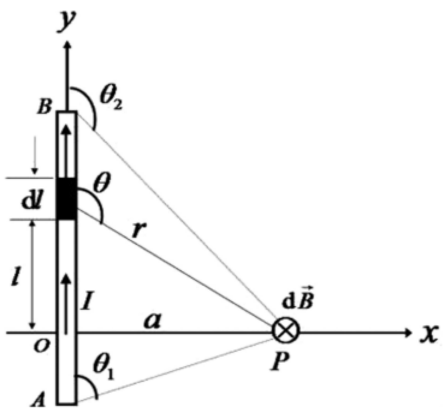


**Fig. 1** Current-carrying direct conductor diagram [5].

### 3.2. Ampère's Circuital Law

The Ampère's circuital law is stated as follows. In a constant magnetic field, the line integral of the magnetic induction along any closed loop L is equal to $\mu_0$ times the algebraic sum of all current intensities across it. It can be expressed by the formula as follows

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \sum_L I \,. \tag{7}$$

The positive and negative values of the current $I$ are specified as follows. When the direction of the current through loop $L$ and the direction around loop $L$ obey the right-hand rule, $I > 0$. Otherwise, $I < 0$. If the current $I$ does not pass through bribe $L$, then it does not contribute to the right end of the above equation. The Ampere loop theorem also has applications in topology

$$\oint_l \boldsymbol{B} \cdot d\boldsymbol{l} = \frac{\mu_0 I}{4\pi} \oint_L \frac{d\boldsymbol{L} \times \boldsymbol{r}}{r^w} \cdot d\boldsymbol{l}. \tag{8}$$

The double loop integral to the right of the above formula is called the connection number integral and is a topological invariant describing the closed curve $\boldsymbol{l}$ and the connection relation. It visually represents the number of times the ampere-loop loops around the current loop. This integral value is always an integer, but may be positive or negative, depending on the orientation of the two loops [6].

Ampère's circuital law is proved by Biot-Savart law, which can also be used to calculate $B$. And The article will show some examples about it. The Ampère's circuital law can be used to find the magnetic field of the current-carrying straight wire (wire thickness cannot be ignored). Suppose the radius of the wire is $R$, and the current $I$ flows uniformly through the cross section. According to the symmetry, the magnitude of magnetic induction $B$ is only related to the vertical distance R of the point drop axis. Article divided it into two situations: $r < R$ and $r > R$.

When $r < R$, only part of the current in the wire passes through loop $L$. This is because that the current density in the wire is $j = \frac{I}{\pi R^2}$ and the area enclosed by the loop is $\pi r^2$. So, the current going through $L$ is $I' = j\pi r^2 = \frac{Ir^2}{R^2}$. After calculation, it arrived that $B = \frac{\mu_0}{2\pi}\frac{rl}{R^2}$ $(r < R)$. The above formula shows that inside the wire, $B$ is proportional to $r$. On the contrary, when $r > R$, $I' = I$, so $B$ can be pushed out $B = \frac{\mu_0 I}{2\pi r}$ $(r > R)$. The above formula shows that viewed from the outside of the wire, the magnetic field distribution is identical to that on the axis of the entire current set $I$.

### 3.3. Gauss's Law

The Gauss's law can be used to find the electric field intensity of a charged body with an inhomogeneous charge. Suppose there is a sphere whose radial is $R$ with an inhomogeneous charge, and the density of the charge $\rho$ at any point in the sphere depends on its distance from the center of the circle $r$, and the scale coefficient is $k$. According to this, there is a formula $\rho = kr$. As can be seen from the formula, the farther away from the center of the circle, the greater $p$. Article divides it into two situations: $r > R$ and $r < R$ [7].

In the first case, it is simple, and the electric field strength on a sphere with the same R is the same. Put the known information into the formula, $E \cdot 4\pi r^2 = \frac{Q_{all}}{\varepsilon_0}$. where $Q_{all}$ is slightly difficult to calculate. Unlike what was discussed in the previous article, the charge distribution here is not uniform and cannot simply be superimposed. So, this article uses calculus to sum $Q_{all}$. In a rectangular coordinate system (see Fig. 2), $Q_{all}$ can be represented as $Q_{all} = \oiiint \rho \cdot dV$.

However, it is very complicated to represent $R$ in rectangular coordinates, which requires three coordinates, resulting in difficult integration. Thus, this article transforms the coordinates and changes the rectangular coordinates to spherical coordinates to make it easy to calculate. For the integral, $dV$ can be converted to $r^2 \sin\varphi \, d\varphi dr d\theta$ [8], leading to

$$Q_{all} = \mathrm{k}\int_0^{\pi}\sin\varphi\, d\varphi \int_0^{2\pi} d\theta \int_0^{R} r^3 dr = k\pi R^4. \tag{9}$$

This means that $E = \frac{kR^4}{4\varepsilon_0 r^2}$. In the other situation, the author still uses Gauss's law to calculate it, which means that $E \cdot 4\pi r^2 = \frac{Q'}{\varepsilon_0}$. Because $r < R$, so the part where $r > R$ makes no contribution to the electric field strength. Thus, $Q' = \mathrm{k}\int_0^{\pi}\sin\varphi\, d\varphi \int_0^{2\pi} d\theta\ \int_0^{r} r^3 dr = k\pi r^4$. One can calculate that $E = \frac{kr^2}{4\varepsilon_0}$.

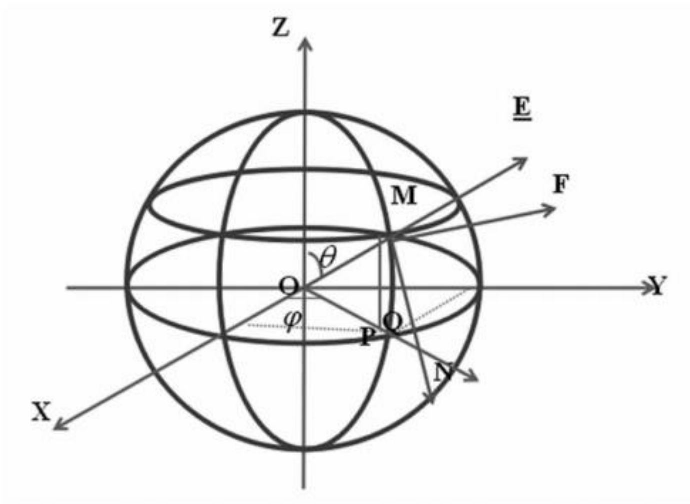


**Fig. 2** Rectangular coordinate system and spherical coordinate system diagram [7].

After looking at the case where $\rho$ and $r$ are proportional, now the author will show case where $\rho$ and $r$ are inversely proportional, i.e., $\rho = \frac{K}{r}$. Likewise, the author divides the region into two situations, $r > R$ and $r < R$.

When $r > R$, the calculation method is the same as the previous one, but the specific representation of the charge density is different. Therefore, the article directly carries out calculation with formulas: $Q_{all} = \mathrm{K}\int_0^{\pi}\sin\varphi\, d\varphi \int_0^{2\pi} d\theta\ \int_0^{R} \mathrm{r}dr = 2k\pi R^2$. Bringing it into the relation $E \cdot 4\pi r^2 = \frac{Q_{all}}{\varepsilon_0}$, it is found that $E = \frac{KR^2}{2\varepsilon_0}\frac{1}{r^2}$. When $r < R$, as article mentioned before, one should change the upper limit of the integral to find $Q'$: $Q' = \mathrm{K}\int_0^{\pi}\sin\varphi\, d\varphi \int_0^{2\pi} d\theta\ \int_0^{r} \mathrm{r}dr = 2k\pi r^2$. Similarly, it is found that $E = \frac{K}{2\varepsilon_0}$ [9].

The calculation of electric field is very important. It plays an important role in studying the routing of energy-aware battery-powered vehicles in networks with charging nodes and contributes to environmental protection. Moreover, it is even used in the non-uniform charge distribution in semiconductor nanoparticle [10].

## 4. Conclusion

In this study, several simple examples of electric and magnetic fields are obtained by using Gauss's law and Biot-Savart's law. Then, through the previous examples, combined with the method of calculus, the intensity of the electric field of the sphere with uneven charge is obtained, and the uniform and uneven examples are compared. In this paper, a method for calculating the electric field intensity of a general charged body with symmetrical charge distribution is given. Through the in-depth understanding and exploration of electric fields, boundaries of electromagnetism can expand,

and make new breakthroughs and progress in energy, communication, medicine, and other fields. Finally, article call on researchers and engineers to continue to explore the application and significance of Gauss's law and Biot-Savart's law to promote development and innovation in the field of electromagnetism. By working together, it is better to understand and utilize electromagnetic phenomena and make greater contributions to the sustainable development and progress of human society.

## References

[1] Álvarez-Gaumé L., Loukas O., Orlando D., and Reffert S. Compensating strong coupling with large charge, JHEP, 2017, 04: 059.

[2] Kimura, T., Goto, T., Shintani, H. et al. Magnetic control of ferroelectric polarization. Nature 2003, 426: 55–58.

[3] Stengel, M., Spaldin, N. & Vanderbilt, D. Electric displacement as the fundamental variable in electronic-structure calculations. Nature Phys 2009, 5: 304–308.

[4] Han Bing, Mi Zhen, Ma Zigang, etc. Research on stator coil field distribution of permanent magnet spherical motor based on Biot-Savart law. Shandong Electric Power, 2020, 47(2): 1-8.

[5] Zhu Zhonghua, Zhang Yuqing, Peng Zhaohui et al. Analysis on the selection of integral arguments in the application of Biot-Savart law. Technology Wind, 2021, 19:43-44.

[6] Tang Jianjian, Li Yiwen, Zhu Junan et al. Topological description of the Ampere loop theorem. College Physics, 2020, 39(02):74-77.

[7] Li Hong, Xu Wenlong. Research on transformation of space Cartesian coordinate system and spherical coordinate system. Journal of Mudanjiang Normal University (Social Sciences), 2017, 23(02):49-51.

[8] S.J. Claessens. Efficient transformation from Cartesian to geodetic coordinates, Computers & Geosciences, 2019, 133: 104307.

[9] S. Pourazarm and C. G. Cassandras. Optimal Routing of Energy-Aware Vehicles in Transportation Networks with Inhomogeneous Charging Nodes. IEEE Transactions on Intelligent Transportation Systems, 2018, 19(8): 2515-2527.

[10] Kozhushner, M. A. Lidskii, B. V. Oleynik, I. I. Posvyanskii, V. S. Trakhtenberg, L. I. Inhomogeneous Charge Distribution in Semiconductor Nanoparticles. The Journal of Physical Chemistry C, 2015, 119(28): 16286–92.