

Enhanced Feynman-Kac Formula for Nonlinear Uncertainty Quantification Problems

He Xu^{1, #}, Xueting Li^{2, #}, Wendi Cai^{3, *, #}

¹ College of Science, Donghua University, Shanghai, China, 201620

² Duke Kunshan University, Kunshan, China, 215316

³ Xi'an Jiaotong-Liverpool University, Suzhou, Jiangsu. 215123

* Corresponding Author Email: moyuhan4077@163.com

#These authors contributed equally.

Abstract. Uncertainty qualification (UQ) issues and partial differential equations (PDEs) are becoming hot topics in the fields of physic and engineering. There are a various number of strategies for handling unknown linear PDEs, including Monte Carlo and the polynomial regression approach. Nevertheless, nonlinear issues could not be effectively solved using existing approach. This paper focuses primarily on a novel method, an enhanced Feynman-Kac formula with a Least Squares Monte Carlo approach. Numerical experiments were conducted in our research to verify its efficacy and convergence. This paper discovers that the enhanced approach exhibits strong performance regarding nonlinear PDE and UQ scenarios with high accuracy and strong performance of convergence. In the future, this technique might be efficiently applied in the fields of engineering and finance in real-world scenarios.

Keywords: Feymann-Kac formula, Uncertainty Quantification, Nonlinear Partial Differential Equation, Stochastic Differential Equation, Least Squares Monte Carlo, Linear Partial Differential Equation, Statistics.

1. Introduction

In physic and engineering realms, numerical modeling and simulation of complex systems is constantly at an astounding rate. The ongoing advancement of numerical methodologies and techniques and the increase in the accessibility of computational resources have enabled this growth. However, the uncertainties may also occur in systems such as uncompleted input data set for inherent variability or a lack of understanding of the actual system [1]. Moreover, it is possible that physical characteristics of systems are unknown, potentially affecting constants that one can bound but cannot precisely identify a priori, perhaps due to impracticality of direct measurements [2]. There are several common approaches to uncertainty quantification issues such as Monte Carlo, perturbation methods, moment equation methods, generalized polynomial chaos and stochastic collocation [3].

When it comes to find solutions to a PDE with uncertainty such as random data can be employed to obtain approximate solutions of this system issue. Considering strong robustness and practicality, Monte Carlo method is the most renowned method at present [4]. Nevertheless, this widely used approach is restricted to solving linear PDE. Pham introduces new versions of the Euler scheme and the walk on spheres method for Monte Carlo calculation of Feynman–Kac representations, as one of methods of valuing options [5].

This paper proposes to introduce Least Squares Monte Carlo (LSMC) with Feynman–Kac representations approach to non-linear PDE with uncertainties and UQ issues. The fundamental tenet of this approach is that, given the cross-sectional data from the simulation, least squares may be used to estimate the conditional expectation of the payout from continuing to maintain the option [6] [7]. The structure of this paper is as followed. The fundamental methodology will be proposed in chapter 2. Then, the improved Feynman-Kac formula with LSMC applied to linear and non-linear PDE will be focused in section 3 with numerical tests in section 4 and 5 respectively.

2. Solutions to UQ Problems based on Polynomial Regression Method

Consider the PDE below:

$$\frac{\partial}{\partial t} u(x, t) + \mu(x, t) \frac{\partial}{\partial x} u(x, t) + \frac{1}{2} * \sigma^2(x, t) \frac{\partial^2}{\partial x^2} u(x, t) - V(x, t) + f(x, t) = 0 \quad (1)$$

Defined for $x \in R$ and $t \in [0, T]$, subject to the terminal condition:

$$u(x, t) = \phi(x) \quad (2)$$

Where T is a parameter, μ, σ, V, ϕ are given functions. In this case, u is the unknown parameter $u : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ Represented by X_t , a stochastic process driven by Brownian motion is defined with the initial condition $X(t) = x$, which satisfies?

$$dX_t = u(x, t)dt + \sigma(x, t)dB_t \quad (3)$$

Thus, the corresponding Feynman-Kac formula offers a solution to the PDE by solving conditional expectations:

$$u(x, t) = E_x \left[\int_t^T e^{-\int_t^s V(X_s, s) ds} f(X_s, s) ds + e^{-\int_t^T V(X_s, s) ds} \phi(X_T) \mid X_t = x \right] \quad (4)$$

SDE Based Regression for Linear Random PDEs performs the regression of linear stochastic PDEs by representing stochastic partial differential equation (SPDE) as the solution of stochastic differential equation (SDE) [8]. The SDE is constructed so that the solution $u(\tau, x)$ of the SPDE at some point $x \in D$ can be expressed as the conditional expectation of some function of the SDE solution. An example of the stochastic representation of deterministic PDEs was introduced as [9]:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, X_0 = x \quad (5)$$

Where a deterministic point, denoted as $x \in \mathbb{R}^d$, is where W is a d -dimensional standard Brownian motion defined on a probability space $(\Omega, F, \mathbb{P})$, with $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ being (suppose) uniformly Lipschitz continuous functions. A Brownian motion with a distinct dimension could be equivalently considered. It is commonly known as the uncertain quantity problem in this work.

This thesis then constructs an SDE containing a deterministic point x , a Brownian motion W , and some boundary conditions. This thesis also introduces the Feynman-Kac formula [10], which maps deterministic data to a set of random variables $x = x(\tau, f, g)$ such that $u(x) = E[x]$, i.e. the expectation of x equals the deterministic solution. When the data is random, the solution $u(\tau, x)$ for random PDE can be expressed as the conditional expectation,

$$u(\tau, x) = E[x \mid \tau, f, g] \quad (6)$$

Where the variance of $u(x)$ can be defined by the variance of x . To determine the value of point x in D , the classical Euler method can be used [11]. Given a finite set of sampling points in D , the approximate expectation of the solution can be determined by solving the regression problem on the orthogonal global polynomial basis, which is

$$u(\cdot) = E[u(\tau, \cdot)] \quad (7)$$

With this approach, this thesis gets a representation of the desired solution field that is globally defined and smooth. This method can be regarded as a combination of sampling and reconstruction techniques, using the classical random solution method to sample point by point, and the global polynomial projection in the sense of least squares. However, this method is a regression method based on stochastic differential equations, which is only applied to find solutions to linear stochastic PDEs with random data and has certain limitations for nonlinear cases [12]. The reason is that solutions of nonlinear PDEs cannot usually be expressed as the

expectation of a random process and are difficult to express in exponential form using the Feynman-Katz formula. For instance, consider a simple non-linear PDE:

$$\frac{\partial u}{\partial t} = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) \quad (8)$$

Where F is a nonlinear function of u and its partial derivatives. For such an equation, it is unrealistic to assume that the solution can be written in the form of the Feynman-Kac formula. In this case, the solution $u(x, t)$ can no longer be expressed as the expectation of a random process. In the following parts, this thesis will focus on our proposed Least Squares Monte Carlo simulation method based on American option pricing.

3. Improved Feymann-Kac Formula for Numerical Solutions of Nonlinear Partial Differential Equations

3.1. The Least Squares Monte Carlo Simulation Method for Pricing American Options

This thesis mainly concentrates on enhancing the Feymann-Kac formula for finding solutions to PDEs based on the Least Squares Monte Carlo simulation method of American option pricing [13]. Firstly, this thesis briefly introduces the process of American option pricing. The pricing process is as follows. This thesis will utilize the Least Squares Monte Carlo simulation method to calculate the American option price at the current time point $t = 0$. M stock price paths are simulated, each starting at $t = 0$ for N Monte Carlo simulations. Each path consists of $N + 1$ trading time points, where the stock price is generated by the formula:

$$S_{i,j} = S_{i,j-1} \times e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z_{i,j}} \quad (9)$$

In the formula, $S_{i,j}$ represents the stock price of the i^{th} path at the j^{th} time point, $S_{i,j-1}$ represents the stock price of the i^{th} path at the $(j - 1)^{th}$ time point, and r representing the rate of return of riskless investment in the market, while σ represents volatility, which is used to measure the degree of asset price fluctuation. The time step Δt indicates the time interval of each simulation step. $Z_{i,j}$ is a random number obeying the standard normal distribution and is used to simulate the random fluctuation of the price path.

This thesis firstly defines the exercise value at the expiration of the option as $V_1 = \max(K - S_{i,N+1}, 0)$, where K represents the exercise price of the option. Then, this thesis discounts V_1 to the previous time point t_N , and obtains the discounted price of V_2 . V_2 is considered as a dependent variable, and the stock value at the time point t_N as an independent variable. The fitting equation $V_2 = f(S_{i,N})$ is obtained by least square fitting. Then, $S_{i,N}$ is re-substituted in the fitting equation f , and V_3 which represents the return value of continued holding is obtained. Comparing the value of V_3 with the value of the return obtained by exercising the option immediately at a point in time before the option expires, this thesis chooses the approach with maximum return, and so on, until $t = 0$, and take the mean outcomes of all paths as $\frac{1}{M} \sum_{i=1}^M V_3$.

The methodology of solving nonlinear PDEs in this thesis is primarily based on the usage of the least squares method in the pricing process: when dividing time points, starting from the value of the last time point, calculating, and using the least squares method for regression fitting correction step by step, and finally obtaining the value of the initial point. In the next section, this thesis will use this idea to improve the Feymann-Kac formula for solving linear PDEs, thereby solving nonlinear PDEs.

3.2. An Improved Approach to Solving Nonlinear Partial Differential Equations Using Feynman-Kac Formula

This thesis considers a common type of PDEs, namely parabolic PDEs. Parabolic PDEs typically describe phenomena such as diffusion or heat conduction in physical systems. This type of equation has second-order derivatives in time and space. Let's take the following equation as an example:

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) + \frac{1}{2} * \frac{\partial^2}{\partial x^2} u(x, t) + f(x, t) = 0 \\ u(x, T) = g(x) \end{cases} \quad (10)$$

In PDEs, $u(x, t)$ represents the state or attribute of the system in space x and time t , called state function or field function. $f(x, t)$ is a known source term or external influence term, which describes the external effect or influence on the system, and $g(x)$ provides us with the boundary value information of the system state at the end of time T . By solving this kind of equations numerically, this thesis can study the evolution rule of the system at different time and space points and provide a basis for further analysis and application.

The Feynman Kac formula is a crucial mathematical tool used to solve PDEs. It links solutions of PDEs with the expected values of stochastic processes, offering a new perspective for solving PDEs. According to the Feynman-Kac formula, in the process of solving PDEs, this thesis can find the solution of a single point at first, then randomly sample a sufficient number of points within a range to solve, and then perform polynomial regression fitting to obtain the solution of the PDE.

For a solution to the single point $u(x_0, t_0)$, this thesis first determines the driving equation according to the given equation:

$$dX_t = dB_t \quad (11)$$

The driving equation of equation (10) is standard Brownian motion and the path starts from X_0 . The simulated path of standard Brownian motion does not stop until the boundary $t = T$ is satisfied, and the time is divided into as small as possible equidistant time intervals. The solution of this point can be obtained by simulating multiple paths and calculating expectations according to Feynman-Kac formula of this equation, where Feynman-Kac formula is:

$$u(x_0, t_0) = E_x \left[\int_{t_0}^T f(x, t) dt + g(x_T) \right] \quad (12)$$

But when f is a nonlinear function of point u , that is, $f = f(x, t, u)$, u is unknown at each point in time, then $\int_{t_0}^T f(x, t) dt$ is also unknown and cannot be estimated at current time point. In Chapter 3.1, the LSMC simulation method for weighted pricing American options is commonly adopted a backward inverse operation [14], which will be employed to solving nonlinear PDEs.

Simulating the path part is consistent with solving the linear partial differential equation problem, and when $t = T$:

$$u(x_T, T) = g(x_T) \quad (13)$$

Using Feynman-Kac formula, $u(x, t)$ of the previous time point is deduced:

$$u(x_{n-1}, t_{n-1}) = \int_{t_{n-1}}^{t_n} f(x_n, t_n, u(x_T, T)) dt + g(x_T) \quad t_n = T \quad (14)$$

Were,

$$\int_{t_{n-1}}^{t_n} f(x_n, t_n, u(x_T, T)) dt \approx f(x_n, x_n, u(x_T, T)) \Delta t \quad (15)$$

$$\begin{aligned} g(x_T) &= u(x_T, T) \\ x_n &= x_T, \quad t_n = T, \quad t_{n-1} = T - \Delta t \end{aligned} \quad (16)$$

After each path is calculated forward once, the least square calculation is performed, with $u(x_{n-1}, t_{n-1})$ as dependent variable and x_{n-1} as independent variable, polynomial regression fitting is performed, and then x_{n-1} is brought into the polynomial to obtain a new $u(x_{n-1}, t_{n-1})$ to replace each path at t_{n-1} . This procedure refers to the Least Squares Monte Carlo simulation method of American option pricing, such as segmentation, reverse operation and polynomial regression fitting strategy.

Continuing forward to calculate:

$$u(x_{n-2}, t_{n-2}) = \int_{t_{n-2}}^{t_{n-1}} f(x_{n-1}, t_{n-1}, u(x_{n-1}, t_{n-1})) dt + g(x_{n-1}) \quad t_{n-1} = T - \Delta t \quad (17)$$

Were,

$$\int_{t_{n-2}}^{t_{n-1}} f(x_{n-1}, t_{n-1}, u(x_{n-1}, t_{n-1})) dt \approx f(x_{n-1}, t_{n-1}, u(x_{n-1}, t_{n-1})) \Delta t \quad (18)$$

$$\begin{aligned} g(x_{n-1}) &= u(x_{n-1}, t_{n-1}) \\ t_{n-2} &= t_{n-1} - \Delta t \end{aligned} \quad (19)$$

Taking $u(x_{n-2}, t_{n-2})$ as dependent variables and x_{n-2} as independent variables, a polynomial regression fitting is performed, and then x_{n-2} is substituted into the polynomial to obtain a new $u(x_{n-2}, t_{n-2})$ to replace the value of each path at x_{n-2} . By analogy, this thesis calculates step by step, and finally bring x_0 into the polynomial fitting with $u(x_0, t_0)$ as dependent variable and x_0 as independent variable and get the solution of the nonlinear partial differential equation at this point. In the same way, enough points are randomly sampled in the required range to solve the problem, and then polynomial regression fitting is carried out, and finally the solution in this interval $u(x, t)$ is obtained.

4. Improved Technique for Solving Nonlinear Partial Differential Equations

4.1. Numerical Experiments

To verify the effectiveness of the improvement method proposed by us, numerical experiments will be conducted on a specific nonlinear PDE. Consider Equation (20):

$$f = -\frac{3}{2} * u^{\frac{1}{3}} * e^{\frac{2}{3}(x+t)} \quad t \in (0,1) \quad (20)$$

$$u(x, 1) = e^{(x+1)} \quad (21)$$

The real solution of this equation is:

$$u(x, t) = e^{(x+t)} \quad t \in (0,1) \quad (22)$$

The time interval $\Delta t = 0.01$ is adopted and 10,000 points are randomly selected from the interval $[0,1]$, with each point simulating 1000 paths. For each path, starting from the boundary conditions of Equation (20) as described in Section 3.2, calculations are carried out step by step using the method described, continuously corrected through regression fitting using the least squares method, ultimately obtaining the solution $u(x, 0)$ for each point. Then, using each point as an independent variable and $u(x, 0)$ as the dependent variable, polynomial regression fitting of order 7 is performed to obtain the solution of $u(x, 0)$ on the interval $[0, 1]$.

Given the real solution:

$$u(x, t) = e^{(x+1)} \quad t = 1 \quad (23)$$

Therefore, the part of f concerning u can be entirely replaced by its real solution, thereby rewriting the nonlinear PDE into a linear stochastic PDE. The linear PDE is resolved. Then, at $t = 0$, the solutions of the nonlinear PDE and the linear PDE on the interval $[0, 1]$, as well as the real solution of the equation, are shown in Figure 1.

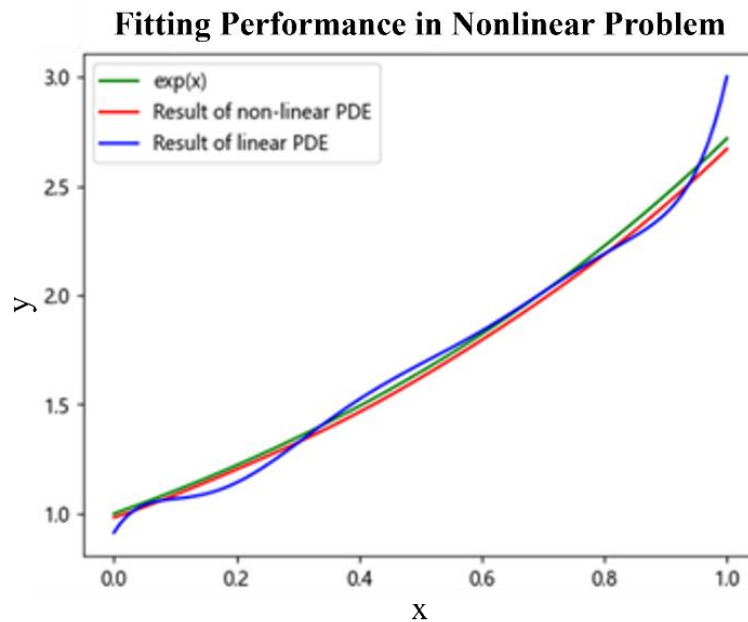


Figure 1. Solutions of Linear PDE and Non-linear PDE Compared to Real Solutions.

Based on Figure 1, it is evident that the solution approach for the nonlinear PDE exhibits superior performance, with results that closely approximate the true solution, surpassing the performance of the solution method for the linear PDE. The average relative error for the solution of the nonlinear PDE problem is 0.006566, while for solving the linear PDE problem is 1.959375. It can be observed that the mean relative error for the nonlinear case is less than that of the linear case, indicating that the solution method for the nonlinear PDE is more efficient than that for the linear PDE.

4.2. Convergence Analysis

The performance of the solution approach is primarily influenced by the time step length, the number of simulation paths, and the degree of regression polynomial fitting. As the time step length decreases, the simulated paths approach the real situation more closely. Increasing the quantity of simulation paths enhances the accuracy of Monte Carlo simulation. The accuracy of the fitting function is also affected by the degree of regression polynomial fitting. Therefore, error convergence analysis is conducted from these three aspects. Below, the sample point $u(1,0)$ is taken to observe the relative error between the solution to the nonlinear PDE equation and the real solution at this point.

The experimental procedure was carried out using technique of controlling variables. Specifically, the number of simulated paths was fixed at 1000 and the polynomial regression order was set to 7 when observing the relationship between time step length and relative error. When analyzing the influence of the quantity of simulated paths on relative error, a time step length of 10^{-3} was set while keeping the polynomial regression order unchanged. Meanwhile, in investigating the effect of polynomial regression order on relative error, the time step length was maintained at 10^{-3} and the quantity of simulated paths was set to 1000. Such settings allowed us to independently observe the impact of each variable change on the results.

From Figure 2 as shown, considering non-linear PDE equations, it can be observed that the relative error drops as the time step Δt drops, the quantity of simulated paths grows, and the polynomial regression order surges exponentially, the relative error gradually decreases, respectively.

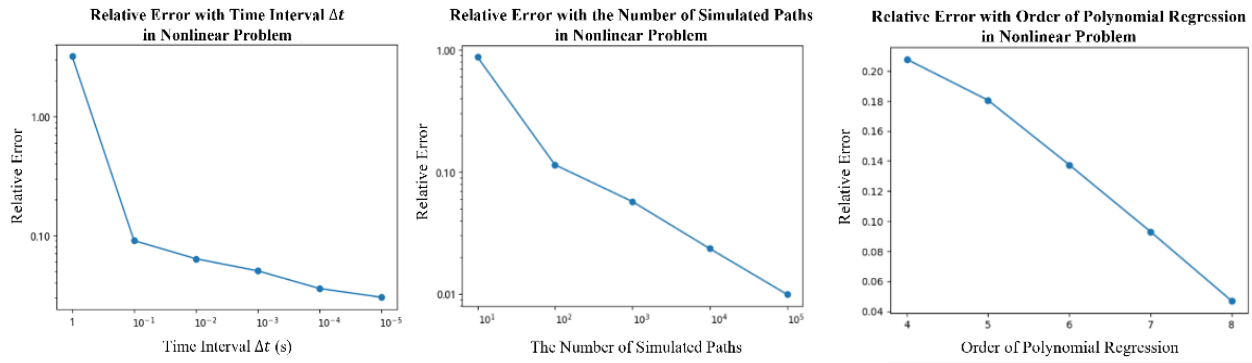


Figure 2. Convergence Analysis in Nonlinear Problem.

Among them, when Δt is decreased to 10^{-2} , the number of simulated paths goes up to 10^2 , the quantity of simulated paths grows to 7, the convergence effect is significantly improved.

5. Improved Method of Solving Nonlinear UQ Problems

5.1. Numerical Experiments

Based on the theory presented in Chapter 2 and methods for solving nonlinear PDEs, the formula (10) is considered in the interval $[0, 1]$.

$$f = (-1) * \left(a_2 + \frac{8}{9} * a_1^2\right) * u^{\frac{1}{3}} * \left(a_0 * e^{\left(\frac{4}{3} * a_1 x + a_2 t + a_3 * a_4\right)}\right)^{\frac{2}{3}} \quad (24)$$

$$u(x, t) = a_0 * e^{\left(\frac{4}{3} * a_1 x + a_2 t + a_3 * a_4\right)} \quad t = 1 \quad (25)$$

Among them, a_0, a_1, a_2, a_3, a_4 are random variables, and the distribution of these parameters is given by specific probability distributions, where a_0 is subjected to a uniform distribution within the interval $[4.5, 4.6]$, while a_1, a_2, a_3, a_4 are respectively subjected to a uniform distribution within the interval $[0, 0.1]$.

The real solution is:

$$u(x, t) = a_0 * e^{\left(\frac{4}{3} * a_1 x + a_2 t + a_3 * a_4\right)} \quad t \in (0, 1) \quad (26)$$

The time interval $\Delta t = 0.01$ is adopted and 100 points are randomly selected from the interval $[0, 1]$, with each point simulating one path. When solving for $u(x_0, 0)$ for each point x_0 , values for a_0, a_1, a_2, a_3, a_4 are randomly chosen according to their distributions once. After obtaining the values, they are substituted into equations (24) and (25), transforming the UQ problem into a nonlinear PDE. Since the driving equation of this equation remains a standard Brownian motion, the solution method is similar to that discussed in Part 3.2 for solving nonlinear PDEs. Eventually, the solution for each point is obtained. Then, using the value of each point as the independent variable and the solution of each point as the dependent variable, a polynomial regression with a regression order of 7 is performed to obtain the average solution to the UQ problem.

Next, part of f regarding u can be replaced entirely with its real solution formula (26), thereby rewriting the stochastic nonlinear PDE into a stochastic linear PDE. This equation can then be solved using methods for solving linear stochastic PDEs. For solving $u(x_0, 0)$ at each point x_0 , a_0, a_1, a_2, a_3, a_4 are randomly sampled once according to their distributions. After each random variable is sampled, the solution of this equation follows the same procedure as described in Section 3.2 for solving linear PDEs. Subsequently, 100 points are randomly selected from $[0, 1]$, and each point's random variables are sampled 100,000 times and plugged into the true solution of the UQ problem equation. The mean is taken, and then polynomial regression fitting is performed to represent the true average numerical solution of the equation. Finally, the real average numerical solution is

compared with solutions to both the linear UQ problem and the nonlinear UQ problem, and the outcomes are presented in Figure 3 as followed.

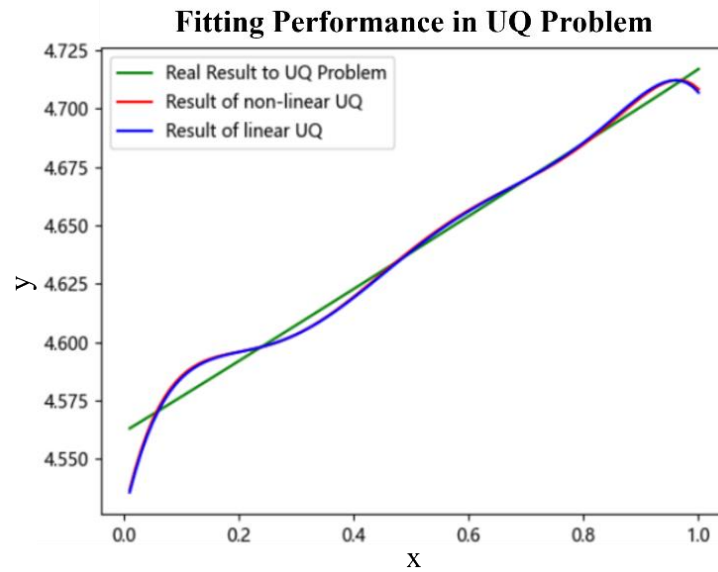


Figure 3. Solutions of Linear UQ Problem and Nonlinear Problem Compared with Real Solutions on Interval [0, 1].

According to Figure 3, the solutions to both the linear and nonlinear UQ problems closely approximate the true solutions, demonstrating a relatively strong solving performance, which also indicates that this method for solving nonlinear PDE problems is effective. The average relative error of solving the nonlinear UQ problem is 0.010061. The average relative error of solving the linear UQ problem is 0.010103, with the mean relative error of the nonlinear being slightly smaller than that of the linear. Overall, the enhanced approach exhibits strong performance in resolving nonlinear UQ issues.

5.2. Convergence Analysis

When it comes to coping with nonlinear UQ problems, the reasons for error generation are the same as in solving nonlinear partial differential problems. Therefore, the analysis of the relative error trend in the solution of the single-point $u(x, 0)$ is carried out similarly, considering parameters such as time step, number of simulation paths, and polynomial regression order. The settings of relevant parameters are consistent with Chapter 3.4.

From Figure 4 as followed, regarding UQ problem, it can be seen that the relative error drops as the time interval Δt drops, the number of simulated paths grows, and the polynomial regression order surges exponentially, the relative error gradually decreases, respectively.

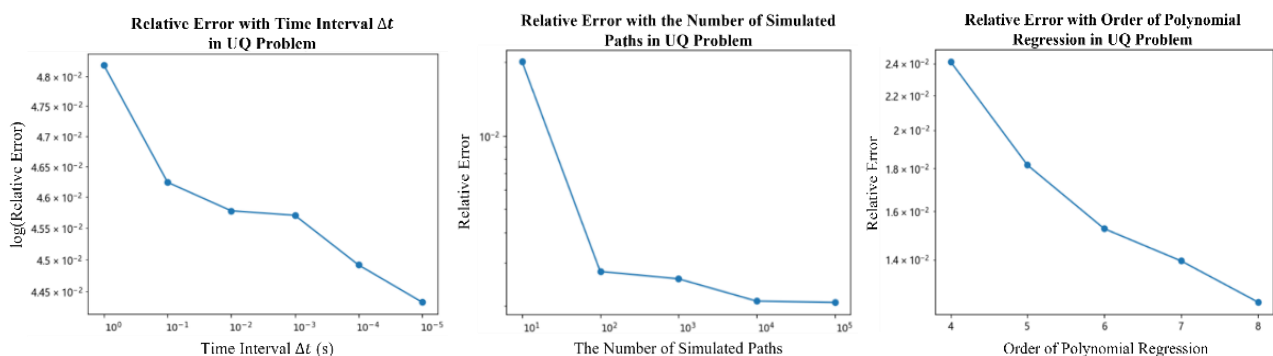


Figure 4. Convergence Analysis in UQ Problem.

Among them, when Δt is reduced to 10^{-1} , the number of simulated paths increases to 10^{-2} , the number of simulated paths increases to 7, and the convergence effect is significantly improved.

6. Conclusions

In this thesis, the enhanced Feynman-Kac formula and the Least Squares Monte Carlo simulation technique for American option pricing are utilized to solve nonlinear PDEs. This thesis verifies the effectiveness of the suggested improvement strategy through numerical experiments. It is discovered that the prioritized method can yield numerical solutions that are extremely similar to the real solution. The error convergence analysis reveals that the relative error of the numerical solution gradually drops, and the convergence effect significantly improves as the time step size decreased, the number of simulated paths increased, and the order of polynomial regression increases. Ultimately, the nonlinear uncertainty quantification (UQ) problem is solved by applying the enhanced approach and combining it with the SDE Based Regression for Linear Random PDEs method. The outcomes of the numerical experiment demonstrate that the suggested approach is effective not only in resolving nonlinear PDEs but also in handling nonlinear UQ issues. This thesis discovered that using the polynomial regression method was prone to dimensional catastrophe at higher orders.

Other approaches can be investigated to enhance the accuracy and efficiency of the numerical solution method while dealing with increasingly complicated nonlinear PDEs and UQ situations. Furthermore, this thesis will support the practical use of the suggested approaches in the sectors of engineering and finance to resolve challenging real-world issues.

References

- [1] Hanges N, Trèves F. On the analyticity of solutions of first-order nonlinear PDE [J]. Transactions of the American Mathematical Society, 1992, 331(2): 627-638.
- [2] Galbally D, Fidkowski K, Willcox K, et al. Non-linear model reduction for uncertainty quantification in large-scale inverse problems [J]. International journal for numerical methods in engineering, 2010, 81(12): 1581-1608.
- [3] Tao Tang and Tao Zhou. (2015), "High-precision numerical methods and theory for uncertainty quantification." Science China Mathematics
- [4] Pham H. Feynman-Kac representation of fully nonlinear PDEs and applications [J]. Acta Mathematica Vietnamica, 2015, 40: 255-269.
- [5] Maire S, Tanré E. Some new simulations schemes for the evaluation of Feynman-Kac representations [J]. 2008.
- [6] Longstaff F A, Schwartz E S. Valuing American options by simulation: a simple least-squares approach [J]. The review of financial studies, 2001, 14(1): 113-147.
- [7] Zhou T, Narayan A, Xu Z. Multivariate discrete least-squares approximations with a new type of collocation grid [J]. SIAM Journal on Scientific Computing, 2014, 36(5): A2401-A2422.
- [8] Smith R C. Uncertainty quantification: theory, implementation, and applications [M]. Society for Industrial and Applied Mathematics, 2013.
- [9] Nouy A. Generalized spectral decomposition method for solving stochastic finite element equations: invariant subspace problem and dedicated algorithms [J]. Computer Methods in Applied Mechanics and Engineering, 2008, 197(51-52): 4718-4736.
- [10] Barles G, Buckdahn R, Pardoux E. Backward stochastic differential equations and integral-partial differential equations[J]. Stochastics: An International Journal of Probability and Stochastic Processes, 1997, 60(1-2): 57-83.
- [11] Mattheij R M M, Rienstra S W, Boonkamp J H M T T. Partial differential equations: modeling, analysis, computation[M]. Society for Industrial and Applied mathematics, 2005.
- [12] Adomian G. A new approach to nonlinear partial differential equations[J]. Journal of Mathematical Analysis and Applications, 1984, 102(2): 420-434.
- [13] Longstaff F A, Schwartz E S. Valuing American options by simulation: a simple least-squares approach[J]. The review of financial studies, 2001, 14(1): 113-147.
- [14] Britten-Jones M, Neuberger A. Option prices, implied price processes, and stochastic volatility[J]. The journal of Finance, 2000, 55(2): 839-866.