Cauchy Mean Value Theorem in Hyperbolic Plane

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Abstract. A Hyperbolic number is a generalization of real number. It is a commutative ring with zero factor generated by two real numbers. It is of great value in practical application. Differential mean value theorems play an important role in real analysis, one of which is Cauchy mean value theorem. By studying the correlative properties of hyperbolic numbers, this paper extends the Cauchy mean value theorem in real analysis to the hyperbolic plane, obtains the hyperbolic mean value theorem, and gives a strict proof. Hyperbolic Cauchy mean value theorem lays a further theoretical foundation for the development of hyperbolic analysis, injects new impetus to hyperbolic analysis, and improves the practical application value of hyperbolic analysis.

Keywords: Hyperbolic number, Mean value theorem, Hyperbolic function, Partial order.

1. Introduction

As hyperbolic geometry, stemming from hyperbolic numbers, continues to evolve, its applications in relativity within physics and its expanding role in the study of complex networks have gained prominence[1]. Notable figures such as Andrei Khrennikov and Gavriel Segre have briefly delved into the analytical framework of hyperbolic spaces[2]. Mutlu Akar has explored the further extension of hyperbolic spaces, broadening their scope to the complex plane[3]. In practical applications, researchers like Dmitri Krioukov have discussed the role of hyperbolic geometry in complex networks[4,5]. Similarly, when considering Euclidean and hyperbolic spaces, one naturally questions whether they share a relationship similar to that found in complex analysis. Consequently, this paper will investigate whether certain analytical theorems from Euclidean spaces apply to hyperbolic spaces.

One of the most typical theorems in Euclidean space is the mean value theorem, including Rohr's mean value theorem, Lagrange's mean value theorem and Cauchy's mean value theorem, which can be proved in a variety of ways [6]. These theorems can be transformed into each other by changing the conditions. Among them, Cauchy's mean value theorem is more simple and convenient in solving problems, and is widely used. It can solve the limit of functions, prove equations and inequalities, prove some fixed-point problems, and prove the monotonicity, continuity and boundness of functions [7]. At the same time, the mean value theorem can be used as a tool in different fields. Rui Guangya used the mean value theorem to study emerging economies [8,9], and Zhang Yuejin et al. applied the mean value theorem to the stereo vision information system [10].

The concepts and methods of hyperbolic geometry play an important role in the mathematical expression of general relativity and in solving physical problems, and are widely used in the theory of gravity in general relativity [11]. In non-Euclidean geometry optics, the path of light propagation will be affected by curvature space, resulting in the path of light propagation different from the straight path in Euclidean space. Hyperbolic geometry, as a kind of non-Euclidean geometry, is widely used to describe and analyze the propagation of light [12]. In addition, hyperbolic geometry has also been applied to the research of multiplex network link prediction and charge controlled memristor simulator [13,14].The importance of hyperbolic geometry and the mean value theorem is self-evident. We have reason to think that extending the mean value theorem to hyperbolic geometry will promote the development of hyperbolic analysis and be widely used in practice.

2. Preliminaries

The set definition of hyperbolic numbers is:

$$\mathrm{D} := \{\varepsilon = x + \mathbf{h}y : x, y \in \mathbb{R}\} \tag{1}$$

$\mathbf{h}$ is a hyperbolic unit $\mathbf{h}$ that satisfies $\mathbf{h}^2 = 1$ and $\mathbf{h} \neq \pm 1$.In the relevant literature, hyperbolic numbers are also known as double complex numbers, space-time complex numbers, split complex numbers, or confused complex numbers. It is known from the knowledge of algebra, ring axioms and commutative properties, that the set D is a commutative ring, and its addition and multiplication operations are given below:

$$\begin{aligned}\varepsilon_1 + \varepsilon_2 &= (x_1 + \mathbf{h}y_1) + (x_2 + \mathbf{h}y_2) \\ &= (x_1 + x_2) + (y_1 + y_2)\mathbf{h}\end{aligned} \tag{2}$$

and

$$\begin{aligned}\varepsilon_1\varepsilon_2 &= (x_1 + \mathbf{h}y_1)(x_2 + \mathbf{h}y_2) \\ &= (x_1x_2 + y_1y_2) + (x_1y_2 + x_2y_1)\mathbf{h}\end{aligned} \tag{3}$$

where $\varepsilon_1 = (x_1 + \mathbf{h}y_1)$ and $\varepsilon_2 = (x_2 + \mathbf{h}y_2) \in \mathrm{D}$. For any hyperbolic number $\varepsilon = (x + \mathbf{h}y) \in \mathrm{D}$,we give a definition: the real part of ε is $\mathrm{Re}(\varepsilon) = x$ and the hyperbolic part of ε is $\mathrm{Im}(\varepsilon) = y$. The conjugate of ε is $\bar{\varepsilon}$, and $\bar{\varepsilon}$ is represented by $\bar{\varepsilon} = x - \mathbf{h}y$. We know that a hyperbolic number ring is a non-commutative bad, also known as a hyperbolic binary number ring or a complex hyperbolic number ring, there are zero factors in the hyperbolic ring of numbers. It can also be expressed as

$$\mathbf{e} := \frac{1+\mathbf{h}}{2} \quad \text{and} \quad \mathbf{e}^{\dagger} := \frac{1-\mathbf{h}}{2} \tag{4}$$

and

$$\mathbf{e}\mathbf{e}^{\dagger} = \frac{1+\mathbf{h}}{2}.\frac{1-\mathbf{h}}{2} = 0 \tag{5}$$

where $\{\mathbf{e}, \mathbf{e}^{\dagger}\}$ is known as the idempotent base of D. In addition, the zero factor is idempotent

$$\mathbf{e}^2 = \mathbf{e} \quad \text{and} \quad (\mathbf{e}^{\dagger})^2 = \mathbf{e}^{\dagger} \tag{6}$$

and we know,

$$\mathbf{e} + \mathbf{e}^{\dagger} = 1 \quad \text{and} \quad \mathbf{e} - \mathbf{e}^{\dagger} = \mathbf{h} \tag{7}$$

They are idempotent zero factors that make up the whole ring. If $\varepsilon = (x + \mathbf{h}y) \in \mathrm{D}$, then idempotent can be expressed as

$$\begin{aligned}\varepsilon &= (x+y)\mathbf{e} + (x-y)\mathbf{e}^{\dagger} = (x+y)(\frac{1+\mathbf{h}}{2}) + (x-y)(\frac{1+\mathbf{h}}{2}) \\ &=: a\mathbf{e} + b\mathbf{e}^{\dagger}\end{aligned} \tag{8}$$

It is easy to see that the zero factor in the set D is a real multiple of $\mathbf{e}$ and $\mathbf{e}^{\dagger}$, Because the real multiples of $\mathbf{e}$ and $\mathbf{e}^{\dagger}$ are on the lines $y = x$ and $y = -x$. As a result, A factor is zero if ε and only if $\varepsilon = x\mathbf{e}$ or $\varepsilon = y\mathbf{e}^{\dagger}$, $x, y \in \mathbb{R} - \{0\}$. In a similar way, for $\varepsilon_1 = a_1\mathbf{e} + a_2\mathbf{e}^{\dagger}$ and $\varepsilon_2 = b_1\mathbf{e} + b_2\mathbf{e}^{\dagger} \in \mathrm{D}$, we also have

$$\varepsilon_1 + \varepsilon_2 = (a_1 + b_1)\mathbf{e} + (a_2 + b_2)\mathbf{e}^{\dagger} \tag{9}$$

$$\zeta_1 \zeta_2 = (a_1 b_1)\mathbf{e} + (a_2 b_2)\mathbf{e}^{\dagger} \tag{10}$$

Through the above analysis and research, we use formula (8) to extend the point (x, y) to the hyperbolic coordinate system, and define the hyperbolic positive region D^{+} and the hyperbolic negative region D^{-}. Figure 1 illustrates the two regions intuitively, with points outside the region treated as hyperbolic numbers that cannot distinguish between positive and negative.

The set

$$\mathrm{D}^{+} := \{\zeta_1 = a\mathbf{e} + b\mathbf{e}^{\dagger} : a \geq 0, b \geq 0\} \tag{11}$$

It is called the set of non-negative hyperbolic numbers (see Figure 1). Therefore, we can give the set of non-positive hyperbolic numbers:

$$\mathrm{D}^{-} := \{\zeta_1 = a\mathbf{e} + b\mathbf{e}^{\dagger} : a \leq 0, b \leq 0\} \tag{12}$$

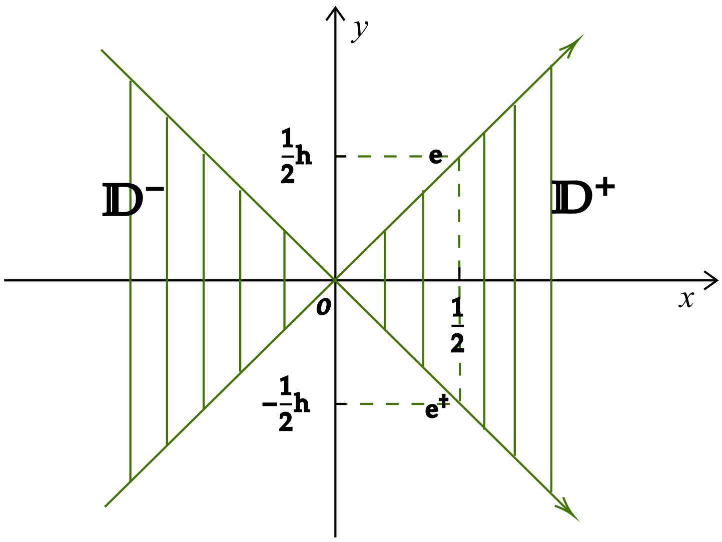

Figure 1: The geometric representation of D^{+} and D^{-}

Obviously, we can know that the following facts are true. if $\zeta_1, \zeta_2 \in \mathrm{D}^{+}$, then $\zeta_1 \zeta_2 \in \mathrm{D}^{+}$, It is not hard to see that D^{+} is closed in multiplication. We know that the size cannot be compared in the field of complex numbers, and it is not convenient to study some problems. In order to better study the problem, we introduce the following relation order in the hyperbolic plane. Therefore, $\zeta, w \in \mathrm{D}$, we have

$$\zeta \preceq w \ \text{ iff } w - \zeta \in \mathrm{D}^{+} \tag{13}$$

and we define that w is D^{-} greater than ζ, or

$$w \preceq \zeta \ \text{ iff } w - \zeta \in \mathrm{D}^{-} \tag{14}$$

and we define that w is D^{-} less than ζ. Similarly, $\zeta \in \mathrm{D}^{+}$ is same as $\zeta \succeq 0$ and that $\zeta \in \mathrm{D}^{+} - \{0\}$ is same as $\zeta \succeq 0$; $\zeta \in \mathrm{D}^{-}$ is same as $\zeta \preceq 0$. For $\zeta_1 = a_1\mathbf{e} + a_2\mathbf{e}^{\dagger}$ and $\zeta_2 = b_1\mathbf{e} + b_2\mathbf{e}^{\dagger}$, we think $\zeta \preceq w$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$.

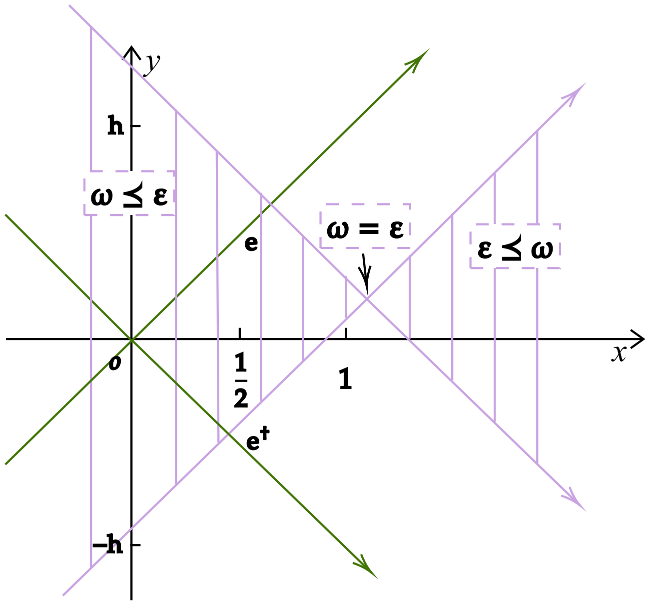

Figure 2: A partial order on $\mathbb{D}$

In Figure 2, ε and ω are points in the hyperbolic plane $\mathbb{D}$. As you can see, the whole plane is divided into empty areas and shaded areas. The shaded region describes the size relationship between the hyperbolic numbers, the point $\varepsilon = \omega$ defines that the two hyperbolic numbers are "equal", the left half plane represents all ω "less than" ε, and the right half plane represents all ω "greater than" ε. The white space represents ω that is not comparable to ε. Through the definition of partial order "$\preceq$" and "$\succeq$", we introduce the concept of size to the hyperbolic plane.

And, the relation $\preceq$ is transitive, reflexive and antisymmetric. Therefore, the relation $\preceq$ represents a partial order in $\mathbb{D}$. We can see this relation order in Figure 2. Let's say that the x and y axes represent one dimensional space. These Spaces are embedded in $\mathbb{D}$, If the speed of light is 1, the zero-factor subset whose real part is positive is called the future direction, and the zero-factor subset whose real part is negative is called the past direction. From the points on lines $y = x$ and $y = -x$, the part where $x > 0$ is the future direction and the part where $x < 0$ is the past direction. The real line is embedded in $\mathbb{D}$ by injection $\varphi : \mathbb{R} \to \mathbb{D}$, For all $a \in \mathbb{R}$,

$$\varphi(a) = \tilde{a} = a\mathbf{e} + a\mathbf{e}^{\dagger} \tag{15}$$

Definition 1 For $\varepsilon = a_1\mathbf{e} + b_1\mathbf{e}^{\dagger}$ and $\omega = a_2\mathbf{e} + b_2\mathbf{e}^{\dagger}$. In $\mathbb{D}$ such that $\varepsilon \preceq \omega$, we define the closed hyperbolic interval $[\varepsilon, \omega]_{\mathbb{D}}$ as

$$[\varepsilon, \omega]_{\mathbb{D}} = \{u \in \mathbb{D} : \varepsilon \preceq u \preceq \omega\}. \tag{16}$$

Similarly, $u = u_1\mathbf{e} + u_2\mathbf{e}^{\dagger} \in [\varepsilon, \omega]_{\mathbb{D}}$ iff u

$$a_1 \le u_1 \le a_2 \quad \text{and} \quad b_1 \le u_2 \le b_2 \tag{17}$$

If $\omega - \varepsilon$ is the non-negative zero-factor hyperbolic number, then the hyperbolic interval $[\varepsilon, \omega]_{\mathbb{D}}$ is said to be degenerate. Moreover, if the positive hyperbolic number $\omega - \varepsilon$ is invertible, then the hyperbolic interval $[\varepsilon, \omega]_{\mathbb{D}}$ is said to be non-degenerate. The hyperbolic interval has the same concept of length as the interval in the real number field. By analogy, the length of any hyperbolic interval is a non-negative hyperbolic number.

Now, consider splitting a function of a complex variable:

$$f: U \subset \Box^{1,1} \to \Box^{1,1} \tag{18}$$

These functions have been studied in many places. Here U is an open subset of $\Box^{1,1}$.

These functions can be written as $f(z) = f_1(x, y) + \mathbf{h} f_2(x, y)$. In the process, We will assume $f_1, f_2 \in C^1(U)$. Now just like in complex analysis, we have to understand what does it mean for the limit of the difference quotient

$$\lim_{\substack{h \to 0 \\ h \notin L}} \frac{f(z_0 + h) - f(z_0)}{h} \tag{19}$$

to exist.

Theorem 2.1 The split complex valued function f is $Cl_{0,1}$-differentiable if and only if

$$f = f_1(u)\mathbf{e} + f_2(v)\mathbf{e}^{\dagger} \tag{20}$$

with this new method for checking differentiability, we are able to check that analogues of some holomorphic functions in $\Box$ are $Cl_{0,1}$-differentiable in $\Box^{1,1}$.

3. Results

Theorem 3.1 (Cauchy mean value theorem for hyperbolic plane)

Let f and g be separated complex variable functions. If f and g are continuous on the closed hyperbolic interval $[\varepsilon, \omega]_D$ and $Cl_{0,1}$-differentiable on the open hyperbolic interval $(\varepsilon, \omega)_D$. And $g(x) = g_1(x_1)\mathbf{e} + g_2(x_2)\mathbf{e}^{\dagger} \neq 0$, then there exist $\zeta = \zeta_1\mathbf{e} + \zeta_2\mathbf{e}^{\dagger}$, $\zeta \in (\varepsilon, \omega)_D$, such that

$$\frac{f'(\zeta)}{g'(\zeta)} = \frac{f(\omega) - f(\varepsilon)}{g(\omega) - g(\varepsilon)} \tag{21}$$

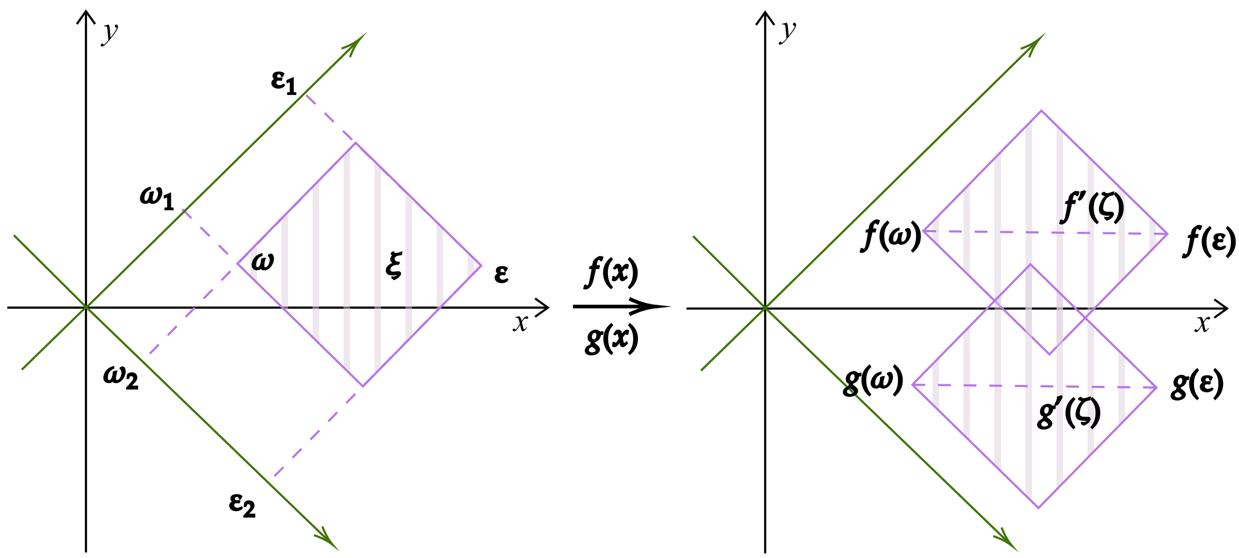

Figure 3: Cauchy's mean value theorem in hyperbolic plane

Proof: we can construct the function

$$F(x) = f(x) - f(\omega) - \frac{f(\omega) - f(\varepsilon)}{g(\omega) - g(\varepsilon)}(g(x) - g(\varepsilon)) \tag{22}$$

According to Theorem2.1, F(x), $f(x)$, and $f(\mathrm{w})$ are separated complex variable functions that can be decomposed into:

$$F(x) = F_1(x_1)\mathbf{e} + F_2(x_2)\mathbf{e}^{\dagger} \tag{23}$$

$$f(x) = f_1(x_1)\mathbf{e} + f_2(x_2)\mathbf{e}^{\dagger} \tag{24}$$

$$f(\mathrm{w}) = f_1(\mathrm{w}_1)\mathbf{e} + f_2(\mathrm{w}_2)\mathbf{e}^{\dagger} \tag{25}$$

Similarly, $g(x), g(\mathrm{w}), g(\mathrm{e})$ can also have the above decomposition, the above decomposition into (22), have

$$F_i(x_i) = f_i(x_i) - f_i(\mathrm{w}_i) - \frac{f_i(\mathrm{w}_i) - f_i(\mathrm{e}_i)}{g_i(\mathrm{w}_i) - g_i(\mathrm{e}_i)}(g_i(x_i) - g_i(\mathrm{e}_i)) \tag{26}$$

We can see that $F_i(\mathrm{w}_i) = F_i(\mathrm{e}_i)$. As a matter of fact, it satisfies Rolle's theorem. there exist $\zeta \in (\mathrm{e}, \mathrm{w})_{\mathrm{D}}$ such that $F_i'(\zeta_i) = 0, F(\zeta) = 0$. Substitution simplifies to: $\frac{f'(\zeta)}{g'(\zeta)} = \frac{f(\mathrm{w}) - f(\mathrm{e})}{g(\mathrm{w}) - g(\mathrm{e})}$.

Figure 3 describes a region $[\mathrm{e}, \mathrm{w}]_{\mathrm{D}}$ in the hyperbolic plane mapped to two regions $[f(\mathrm{w}), f(\mathrm{e})]_{\mathrm{D}}$, $[g(\mathrm{w}), g(\mathrm{e})]_{\mathrm{D}}$ under the mapping of f and g, respectively. The inner point $\zeta \in (\mathrm{e}, \mathrm{w})_{\mathrm{D}}$ maps to the two values $f'(\zeta)$ and $g'(\zeta)$ under the mapping of $f'(x)$ and $g'(x)$, respectively. Figure 3 reflects the geometric meaning of Cauchy's mean value theorem.

4. Conclusions

In this paper, by studying the basic properties of hyperbolic numbers, the hyperbolic mean value theorem is extended to hyperbolic space, and the geometric meaning of the hyperbolic mean value theorem is described. Hyperbolic Cauchy mean value theorem will have great application value, not only can be used as a powerful proof tool to promote the development of hyperbolic analysis and lay a foundation for the theoretical study of hyperbolic analysis. Moreover, the hyperbolic Cauchy mean value theorem, as a basic property in non-Euclidean space, can further improve the application value of hyperbolic analysis in practice, and inject impetus into the practical application of hyperbolic analysis.

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