The First Mean Value Theorem of integral in Perplex Plane

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Abstract. Perplex numbers are an extension of the real numbers, consisting of pairs of real numbers that form a commutative ring. This article considers the partial order structure of perplex numbers and discusses the concept of perplex interval. Based on the definition of integration of perplex functions of perplex variable, this article we investigate the first mean value theorem of integral in perplex plane which can significantly enrich the mathematical toolbox by providing essential methods for estimating and approximating perplex number functions. It also simplifies the solution of complex integral problems involving perplex numbers and fosters theoretical extensions by exploring the behavior of perplex number functions under various conditions. This, in turn, promotes the development of perplex calculus, enhancing the overall theoretical and practical framework of perplex number analysis. In physics, particularly in relativity, studying the first mean value theorem for perplex numbers can provide more precise methods for describing and estimating physical quantities, enhancing the application of hyperbolic numbers in these fields.

Keywords: Perplex Number, The First Mean Value Theorem of Integral, Perplex Interval.

1. Introduction

Perplex numbers also called hyperbolic number are commutative rings with zero factors generated by two real numbers. Dating back to 1986, P. Fjelstad [1] introduced perplex numbers to extend special relativity. After this, the hyperbolic plane was also studied, for example in [2]. Starting with the definition of perplex number , many properties of perplex analysis, for example in[3,4].

M.E. Luna-Elizarrarás et al. [5] revealed the partial order structure of hyperbolic numbers. In 2016, A. S. Balankin et al.[6] defined the hyperbolic intervals. In 2020, Saini et al. [3]discuss some properties of linear functionals on topological hyperbolic and topological bicomplex modules. In 2022, M. E. Luna-Elizarrarás [7] introduced the integration of functions of hyperbolic variable.

Some progress has also been made in application of perplex number. Tellez G. [8] extend shannon entropy and the chaos game algorithm to perplex numbers plane. In 2021, Soykan [9] further developed the concept of generalized hyperbolic Jacobi Stahl-Lucas numbers to investigate their characteristics and examine how they are connected to hyperbolic Jacobi Stahl-Lucas numbers.

In calculus, the mean value theorem of integrals provides an effective way to understand relationships between function values and their integral, for example in[10-12],

Based on the above research, this paper proves the first mean value theorem of integral in perplex plane. This will give impetus to the theory of integration in perplex analysis.

2. Preliminaries

The set of perplex numbers $\mathbb{P}$ also called split complex numbers is defined as

$$\mathbb{P} = \{c = a + bh | a, b \in \mathbb{R} \ , \ h^2 = 1\} \tag{1}$$

Consider the addition and multiplication of perplex numbers, for any $c_1 = a_1 + b_1 h$ and $c_2 = a_2 + b_2 h \in \mathbb{P}$:

$$\begin{aligned} c_1 + c_2 &= (a_1 + b_1 h) + (a_2 + b_2 h) \\ &= (a_1 + a_2) + (b_1 + b_2) h \end{aligned} \tag{2}$$

and

$$c_1c_2 = (a_1 + b_1h)(a_2 + b_2h)$$
$$= (a_1a_2 + b_1b_2) + (a_1b_2 + a_2b_1)h \tag{3}$$

Let $$h_+ = \frac{1+h}{2}\ ,\ h_- = \frac{1-h}{2}$$

$$\text{then } h_+h_- = 0 \tag{4}$$

$$h_+^2 = h_+ \tag{5}$$

$$h_-^2 = h_- \tag{6}$$

and

$$c = a + bh = (a+b)\frac{1+h}{2} + (a-b)\frac{1-h}{2} \quad \text{for any } c = a + bh \in \mathbb{P}.$$

Thus $h_+ = \frac{1+h}{2}, h_- = \frac{1-h}{2}$ is the base of $\mathbb{P}$ and every perplex numbers can also be expressed as $uh_+ + vh_-$ where $u, v \in \mathbb{R}$.

Addition and multiplication operations of the perplex numbers also have definitions in these new coordinates.

For$c_1 = u_1h_+ + v_1h_-, c_2 = u_2h_+ + v_2h_- \in \mathbb{P}$

$$c_1 + c_2 = (u_1 + u_2)h_+ + (v_1 + v_2)h_- \tag{7}$$

$$c_1c_2 = (u_1u_2)h_+ + (v_1v_2)h_- \text{[14],[15]}. \tag{8}$$

Definition 1: The non-negative perplex number and non-positive perplex number (see Figure 1)is defined as following:

$$\mathbb{P}^+ = \{c = uh_+ + vh_- | u \geq 0, v \geq 0\} \tag{9}$$

and

$$\mathbb{P}^- = \{c = uh_+ + vh_- | u \leq 0, v \leq 0\} \tag{10}$$

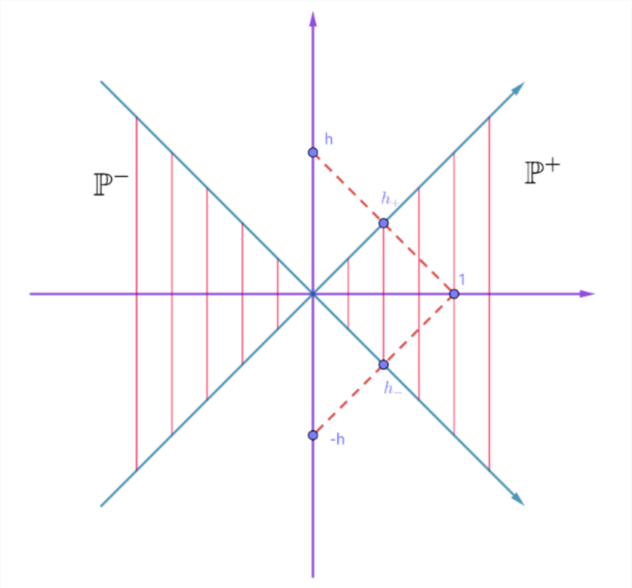


Figure 1. the sets $\mathbb{P}^+$ and $\mathbb{P}^-$

Definition 2[16]:If $u_1 \geq u_2$ and $v_1 \geq v_2$ which means $c_1 - c_2 \in \mathbb{P}^+$□ ,we say that c_1 is $\mathbb{P}$ -greater than c_2 noted by $c_1 \succcurlyeq c_2$. If $u_1 \leq u_2$and $v_1 \leq v_2$ which means $c_1 - c_2 \in \mathbb{P}^-$,we say that c_1 is $\mathbb{P}$□-greater than c_2 noted by $c_1 \preccurlyeq c_2$.

Definition 3[16]: closed perplex interval $[c_1, c_2]_{\mathbb{P}}$ (see Figure 2)is defined by

$$[c_1, c_2]_{\mathbb{P}} = \{c \in \mathbb{P} | c_1 \preccurlyeq c \preccurlyeq c_2\}$$

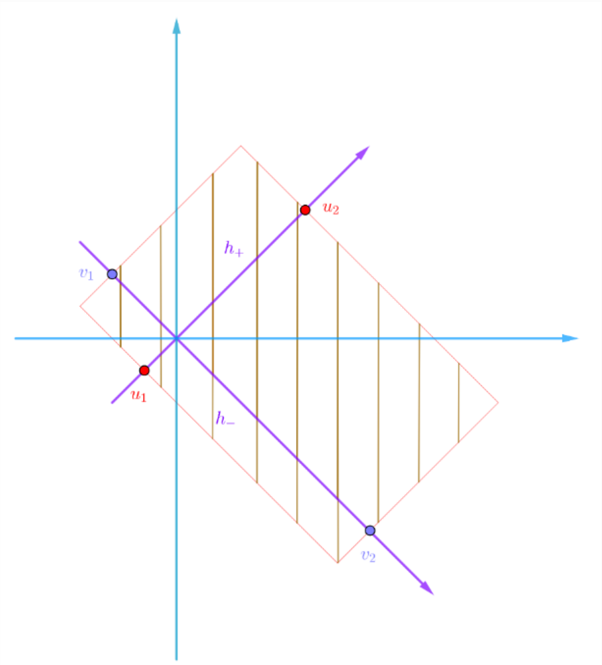


Figure 2. The perplex interval $[c_1, c_2]_{\mathbb{P}}$

The relevant concepts as following:

(i) $c \in \mathbb{P}^+$ if and only if $c \succcurlyeq 0$.
(ii) $c \in \mathbb{P}^-$ if and only if $c \preccurlyeq 0$.
(iii) $c \in \mathbb{P}^+ - \{0\}$ if and only if $c \succ 0$.
(iv) $c \in \mathbb{P}^- - \{0\}$ if and only if $c \prec 0$.

Definition 4: $m = \inf_{c \in [c_1, c_2]_{\mathbb{P}}} f(c)$ is the infimum of $f(c)$ if the followings hold.

(i) For any $c \in [c_1, c_2]_{\mathbb{P}}$, we have that $m \preccurlyeq f(c)$.
(ii) For every $\varepsilon_1, \varepsilon_2 > 0$, we have that $m \succcurlyeq f(c) - c_\varepsilon$ where $c_\varepsilon = \varepsilon_1 h_+ + \varepsilon_2 h_-$.

Definition 5: $M = \sup_{c \in [c_1, c_2]_{\mathbb{P}}} f(c)$ is the supremum of $f(c)$ if the followings hold.

(i) For any $c \in [c_1, c_2]_{\mathbb{P}}$, we have that $M \succcurlyeq f(c)$.
(ii) For every $\varepsilon_1, \varepsilon_2 > 0$, we have that $M \preccurlyeq f(c) + c_\varepsilon$ where $c_\varepsilon = \varepsilon_1 h_+ + \varepsilon_2 h_-$.

Define $f(c) = f_1(a, b) + f_2(a, b)h$ is a perplex valued function on $[c_1, c_2]_{\mathbb{P}}$.

Lemma 1 [17]: The perplex valued function f is $C\ell_{0,1}$-differentiable at c_0 if and only if

$$\frac{\partial f_1}{\partial a} = \frac{\partial f_2}{\partial b} \text{ and } \frac{\partial f_2}{\partial a} = \frac{\partial f_1}{\partial b} \tag{11}$$

Lemma 2 [17]: If f is $C\ell_{0,1}$-differentiable function in $\mathbb{P}$, then $f(c)$ can also be expressed as $f_{h_+}(u)h_+ + f_{h_-}(v)h_-$ where $f_{h_+}, f_{h_-}: \mathbb{P} \to \mathbb{P}$ and $f: \mathbb{P} \times \mathbb{P} \to \mathbb{P}$.

Then the perplex integration of $C\ell_{0,1}$-differentiable function f on $[c_1, c_2]_{\mathbb{P}}$ is defined as

$$I = \int_{c \in [c_1, c_2]_{\mathbb{P}}} f(c) \otimes dc \quad dc = (da, db) \tag{12}$$

Where $c = a + bh$.

According to [11] and lemma 1, the integration of $f(c)$ on $[c_1, c_2]_{\mathbb{P}}$ is also

$$\int_{c \in [c_1, c_2]_{\mathbb{P}}} f(c) \otimes dc = h_+ \int_{u_1}^{u_2} f_{h_+}(u) du + h_- \int_{v_1}^{v_2} f_{h_-}(v) dv \tag{13}$$

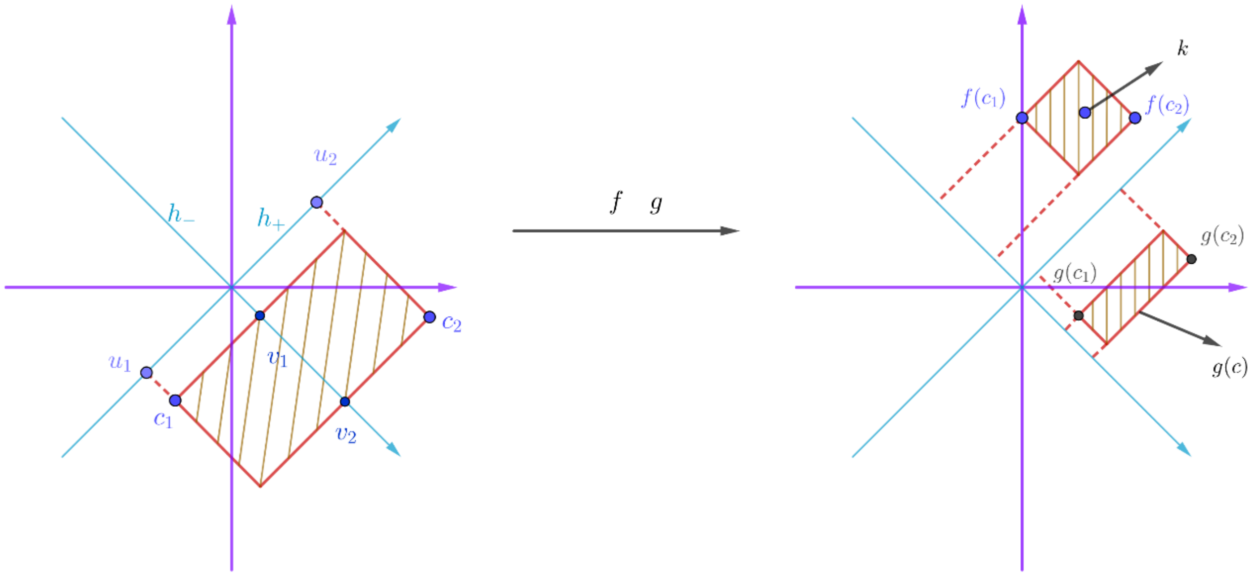
Where $c = uh_+ + vh_-$.

3. First mean-value theorem for the integral in $\mathbb{P}$

Theorem(First mean-value theorem for the perplex integrals in $\mathbb{P}$):Let the function f, g is $C\ell_{0,1}$-differentiable over the interval $[c_1, c_2]_{\mathbb{P}}$,$m = inf_{c\in[c_1,c_2]_{\mathbb{P}}} f(x)$,$M = sup_{c\in[c_1,c_2]_{\mathbb{P}}} f(x)$. If $g(c) \succcurlyeq 0$ holds for any $c \in [c_1, c_2]_{\mathbb{P}}$, then

$$\int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)] \otimes dc = k \int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c) \otimes dc \tag{14}$$

Where $k \in [m, M]_{\mathbb{P}}$.



Proof: Firstly we prove that

$$mg(c) \preccurlyeq f(c)g(c) \preccurlyeq Mg(c) \tag{15}$$.

Let $m = u_1 h_+ + v_1 h_-$, $M = u_2 h_+ + v_2 h_-$ and $f(c) = f_{h_+}(u)h_+ + f_{h_-}(v)h_-$, $g(c) = g_{h_+}(u)h_+ + g_{h_-}(v)h_-$.

Then we have that $u_1 \le f_{h_+}(u) \le u_2$,$v_1 \le f_{h_-}(u) \le v_2$and $g_{h_+}(u) \ge 0$,$g_{h_-}(u) \ge 0$.

Thus

$$\begin{aligned} f(c)g(c) &= (f_{h_+}(u)h_+ + f_{h_-}(v)h_-)(g_{h_+}(u)h_+ + g_{h_-}(v)h_-) \\ &= f_{h_+}(u)g_{h_+}(u)h_+ + f_{h_-}(v)g_{h_-}(v)h_- \\ &\precsim u_1 g_{h_+}(u)h_+ + v_1 g_{h_-}(v)h_- \\ &= Mg(c) \end{aligned}$$

And

$$\begin{aligned} f(c)g(c) &= (f_{h_+}(u)h_+ + f_{h_-}(v)h_-)(g_{h_+}(u)h_+ + g_{h_-}(v)h_-) \\ &= f_{h_+}(u)g_{h_+}(u)h_+ + f_{h_-}(v)g_{h_-}(v)h_- \\ &\succsim u_2 g_{h_+}(u)h_+ + v_2 g_{h_-}(v)h_- \\ &= mg(c) \end{aligned}$$

Therefore

$$mg(c) \preccurlyeq f(c)g(c) \preccurlyeq Mg(c) \tag{16}$$

We next prove that

$$m \int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c) \otimes dc \preccurlyeq \int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)] \otimes dc \preccurlyeq M \int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c) \otimes dc \tag{17}$$

Since

$$\int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)]\otimes dc = \int_{c\in[c_1,c_2]_{\mathbb{P}}} [(f_{h_+}(u)h_+ + f_{h_-}(v)h_-)(g_{h_+}(u)h_+ + g_{h_-}(v)h_-)]\otimes dc$$
$$= \int_{c\in[c_1,c_2]_{\mathbb{P}}} [f_{h_+}(u)g_{h_+}(u)h_+ + f_{h_-}(v)g_{h_-}(v)h_-]\otimes dc$$
$$= h_+\int_{u_1}^{u_2} f_{h_+}(u)g_{h_+}(u)du + h_-\int_{v_1}^{v_2} f_{h_-}(v)g_{h_-}(v)dv$$

We have that

$$\int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)]\otimes dc = h_+\int_{u_1}^{u_2} f_{h_+}(u)g_{h_+}(u)du + h_-\int_{v_1}^{v_2} f_{h_-}(v)g_{h_-}(v)dv$$
$$\succcurlyeq h_+\int_{u_1}^{u_2} mg_{h_+}(u)du + h_-\int_{v_1}^{v_2} mg_{h_-}(v)dv$$
$$= h_+\int_{u_1}^{u_2} (u_1h_+ + v_1h_-)g_{h_+}(u)du + h_-\int_{v_1}^{v_2} (u_1h_+ + v_1h_-)g_{h_-}(v)dv$$
$$= (u_1\int_{u_1}^{u_2} g_{h_+}(u)du)h_+ + (v_1\int_{v_1}^{v_2} g_{h_-}(v)dv)h_-$$
$$= m\int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c)\otimes dc$$

And

$$\int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)]\otimes dc = h_+\int_{u_1}^{u_2} f_{h_+}(u)g_{h_+}(u)du + h_-\int_{v_1}^{v_2} f_{h_-}(v)g_{h_-}(v)dv$$
$$\preccurlyeq h_+\int_{u_1}^{u_2} Mg_{h_+}(u)du + h_-\int_{v_1}^{v_2} Mg_{h_-}(v)dv$$
$$= h_+\int_{u_1}^{u_2} (u_2h_+ + v_2h_-)g_{h_+}(u)du + h_-\int_{v_1}^{v_2} (u_2h_+ + v_2h_-)g_{h_-}(v)dv$$
$$= (u_2\int_{u_1}^{u_2} g_{h_+}(u)du)h_+ + (v_2\int_{v_1}^{v_2} g_{h_-}(v)dv)h_-$$
$$= M\int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c)\otimes dc$$

Therefore

$$m\int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c)\otimes dc \preccurlyeq \int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)]\otimes dc \preccurlyeq M\int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c)\otimes dc \quad (18)$$

Let $k\in[m,M]_{\mathbb{P}}$

$$\int_{c\in[c_1,c_2]_{\mathbb{P}}} [f(c)g(c)]\otimes dc = k\int_{c\in[c_1,c_2]_{\mathbb{P}}} g(c)\otimes dc \quad (19)$$

4. Conclusions

This article investigates the theorem of integral in perplex plane. Perplex numbers are commutative rings with zero factors generated by two real numbers. This article gets the first mean value theorem of integrals in perplex plane, which will provide the basis for the further application of integral theory in perplex plane. It provides a powerful method for dealing with the integration, estimation and approximation of perplex number functions. The behavior characteristics of perplex number function under different conditions can be discussed, and the theoretical framework of hyperbolic number analysis can be enriched. What's more, perplex numbers play an important role in modeling relativistic effects such as Lorentz transformations and relativistic velocity superpositions. By applying the first mean value theorem of integrals, these transformations can be estimated more accurately, leading to more precise predictions and calculations in relativistic dynamics.

References

[1] Fjelstad P. Extending special relativity via the perplex numbers [J]. American Journal of Physics, 1986, 54: 416-422.

[2] Sobczyk G. The Hyperbolic Number Plane [J]. The College Mathematics Journal, 1995, 26: 10.2307/2687027.

[3] Saini H., Sharma A., Kumar R. Some Fundamental Theorems of Functional Analysis with Bicomplex and Hyperbolic Scalars [J]. Advances in Applied Clifford Algebras, 2020, 30: 10.1007/s00006-020-01092-6.

[4] Kumar R., Saini H. Topological Bicomplex Modules [J]. Advances in Applied Clifford Algebras, 2016, 26: 10.1007/s00006-016-0646-1.

[5] Luna-Elizarrarás, M. & Shapiro, Michael & Struppa, Daniele & Vajiac, Adrian. (2015). Bicomplex Holomorphic Functions. 10.1007/978-3-319-24868-4.

[6] Balankin A.S., Bory-Reyes J., Luna-Elizarrarás M.E., Shapiro M. Cantor-type sets in hyperbolic numbers [J]. Fractals, 2016, 24(04).

[7] Luna-Elizarrarás M., Fangfang. Integration of Functions of a Hyperbolic Variable; Research on power load forecasting based on Improved BP neural network [J] [D]. Complex Analysis and Operator Theory; Harbin Institute of Technology, 2022; 16: 10.1007/s11785-022-01197-9, 2011.

[8] Tellez G., Bory-Reyes J. Extensions of the Shannon Entropy and the Chaos Game Algorithm to Hyperbolic Numbers Plane [J]. Fractals, 2021, 29: 1-15. DOI: 10.1142/S0218348X21500134.

[9] Soykan Y., Taşdemir E. A study on hyperbolic numbers with generalized Jacobsthal numbers components [J]. The International Journal of Nonlinear Analysis and Applications (IJNAA), 2022: 1-17. DOI: 10.22075/ijnaa.2021.22113.2328.

[10] Narita K., Endou N., Shidama Y. The First Mean Value Theorem for Integrals [J]. Formalized Mathematics, 2008, 16: 51-55. DOI: 10.2478/v10037-008-0008-0.

[11] Zhou Y. Mean Value Theorem and Its Uses [J]. Highlights in Science, Engineering and Technology, 2023, 72: 926-930. DOI: 10.54097/48pkqt70.

[12] Wu S. Proof and Application of the Mean Value Theorem [J]. Highlights in Science, Engineering and Technology, 2023, 72: 565-571. DOI: 10.54097/nw2nd028.

[13] Cakir H. Algebraic and geometric applications of hyperbolic numbers and hyperbolic number matrices [D]. Akdeniz University, 2017: 6.

[14] Nešović E., Petrović-Torgašev M. Some trigonometric relations in the Lorentzian plane [J]. Kragujevac J. Math, 2003, 33(25): 219-225.

[15] Öztürk İ., Özdemir M. Affine transformations of hyperbolic number plane [J]. Boletín de la Sociedad Matemática Mexicana, 2022, 28: 10.1007/s40590-022-00455-2.

[16] Emanuello J., Nolder C. Projective Compactification of R-1,R-1 and Its Mobius Geometry [J]. Complex Analysis and Operator Theory, 2015, 9: 329-354. DOI: 10.1007/s11785-014-0363-5.

[17] Agarwal R. Bicomplex Polygamma Function [J]. Tokyo Journal of Mathematics, 2007, 30: 10.3836/tjm/1202136693.