

The Application of Stochastic Processes in Options Pricing Model

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Abstract. Since English botanist Brown first observed the inconsistent mobility of pollen floating in liquid in 1827, stochastic motion has become one of the primary research topics today, and the use of stochastic processes in financial markets has never stopped [1]. The change in the process in any time interval do not affect the change in any other time intervals that do not overlap with of Brownian Motion, which possesses stationary independent increments. Therefore, this paper obtains the Black-Scholes Merton formula from the stochastic process through layers of derivation and analyses its limitations in terms of its existence. Therefore, this paper will derive the BSM formula and the logical relationships between them, step by step, starting from the stochastic process and discuss the problems and limitations of the BSM formula. The experiential results demonstrate that the proposed method can achieve better performance.

Keywords: Stochastic Processes, Brownian Motion, Itô calculus, Black-Scholes Differential Equation.

1. Introduction

Anyone interested in quantitative investment is aware of the Black-Scholes Merton formula for quantitative modelling of stock prices, option prices, and other derivative prices. However, the BSM formula is just a result. If we look up the derivation system of the BSM formula on search engines, we will see, "Brownian motion", "Ito's Lemma", "stochastic differential equation" these concepts, which are interlocked to form this complete analytical system. Therefore, in this paper, we will derive the BSM formula and the logical relationships between them, step by step, starting from the stochastic process and discuss the problems and limitations of the BSM formula.

2. Stochastic Process

A measurable function $X: [0, \infty) \times \Omega \rightarrow \mathbb{R}$ is a stochastic process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ [1]. This represents a set of events that occur sequentially in time or space for a random variable. A continuous stochastic process is one in which the random variable is continuous over the whole space. Otherwise, a discrete stochastic process is one in which the random variable is only defined at some discrete points.

3. Brownian Motion

Robert Brown, a botanist, is honored with the name of the Brownian motion after his first discovery that little pollen grains floating in water exhibit an extremely erratic and unexpected state of motion [1]. Ovidiu Calin provides a thorough explanation of the Brownian motion, which is formalized as follows:

Brownian motion is a stochastic process that meets the following conditions:

- 1) The process begins at the origin point, $B_0=0$;
- 2) The increments in the Brownian process are stationary and independent;
- 3) The Brownian process is continuous in t ;
- 4) The increases in the process follows normal distribution, $B_t - B_s \sim N(0, |t-s|)$ [2].

All characteristics of a Brownian motion $B(t)$ that begins at x are present in the process $X_t = x + B_t$. $B_t - B_s$ is stationary, hence the only factor affecting its distribution function is the time interval $t - s$.

3.1. Brownian Motion is a Markov Process

A process is a Markov process if it is stochastic, the historical values of the variables do not affect the forecast of the future value, and the future value prediction solely affects the present value.

If the distribution of x_{k+1} conditioned on F_k is identical to the distribution of x_{k+1} conditioned on x_k for each $k = 0, 1, \dots, n - 1$, this is known as the Markov property [3].

The change in the process in any time interval do not affect the change in any other time intervals that do not overlap with it, according to Condition 2 of Brownian Motion, which possesses stationary independent increments. The property states that Brownian motion is a Markov process, meaning that every position after time t is exclusively relevant to that position and has no influence on the past historical trajectory. In other words, all the information necessary to forecast the process's future is reflected in its present value.

3.2. Wiener Process

It wasn't until 1918 that Norbert Wiener, an American mathematician, came up with a mathematical model for Brownian motion, known as a Wiener process. If a process is Markov and the anticipated value of the variable change per unit time follows a normal distribution with $E(x) = 0$, $Var = 1$, then it is a Wiener process. At the same time, we give the definition of generalized Wiener process: When two processes are appended, the resultant process is also a Wiener process if one of the processes is a Wiener process and the other is a process with constant drift rate [1].

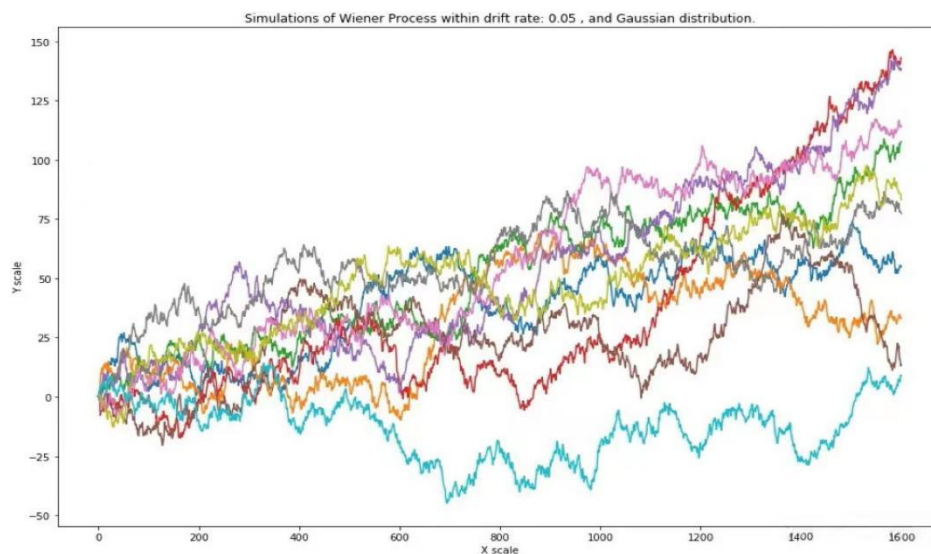


Figure 1. Wiener process simulation

This is the one-dimensional case of Brownian motion. Where drift rate is 0, it represents the general Wiener process. Let's change the code from $N=10$ to $N=1$, which is a single Wiener process.

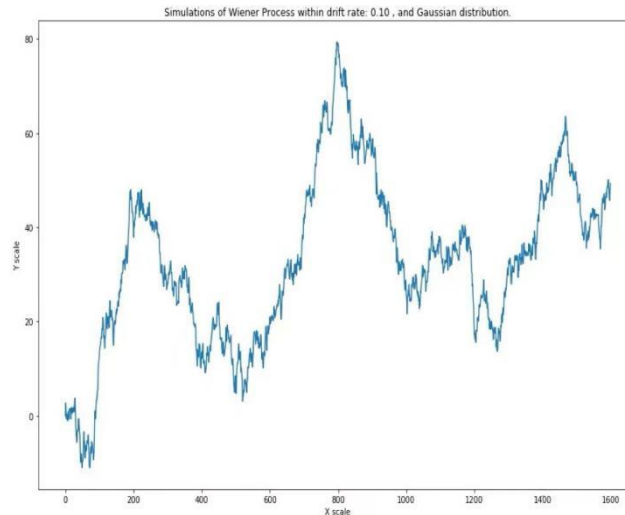


Figure 2. Simulation of Wiener Process within draft rate 0.1

The single Wiener process looks very similar to the behavior of the stock price curve, so people are interested in using it to describe the behavior of the stock price. Indeed, in 1900 a Frenchman called Louis Bachelier was using Brownian motion to predict the prices of stocks and others in doctoral dissertation, Bachelier put forward an explicit description of the price fluctuations of the underlying investment commodity: The probability model is unable to precisely anticipate all the factors that influence the rise and fall of the stock price. However, a mathematical model may be developed to examine the likelihood of the market's rise and fall, that is, to provide a stochastic model of the operation of the stock price based on Brownian motion, at a certain static moment of the market [1]. The idea is fascinating: instead of directly predicting the price of a traded object, differential equations are used to express the rate of changes of price. However, Louis Bachelier did not attract much attention at that time, but the clever intervention of Brownian motion makes it possible to characterize asset prices using stochastic processes.

3.3. The Property of Brownian Motion

Regarding using Brownian motion and its variants to represent stock values, Brownian motion has several fascinating features that have significant ramifications. These properties consist of:

- 1) The trajectory of Brownian motion will oscillate up and down the timeline t frequently.
- 2) The movement of $B(t)$ won't deviate too far, which limits from plus or minus one standard deviation at any given time t .
- 3) Make $M(t)$ be the maximum value that can be reached between time 0 to time t , therefore, $M(t) = \max B(t) (0 \leq s \leq t)$ and $\text{Prob}(M(t) \geq a) = 2\text{Prob}(B(t) \geq a)$. Therefore, $M(t) = \max B(t) (0 \leq s \leq t)$ and $\text{Prob}(M(t) \geq a) = 2\text{Prob}(B(t) \geq a)$. (a is any given threshold value)
- 4) Brownian motion $B(t)$ is continuous, but the function itself is nowhere differentiable [2].

For the first two properties. The following diagram gives the values of time 0 to t for a sample of 15 standard Brownian motion trajectories. Although they exhibit their own randomness, each trajectory reciprocally traverses the line $y = 0$ (the time axis t), with only few sample trajectories with a one-sided oscillation in the line $y = 0$. Moreover, the black parabola is the curve of equation $t = y^2$. None of the sample trajectories will diverge significantly from the distance $B(0) \pm \sqrt{t}$ on this parabola, even though each one contains enough randomness in time t . In the right-side, the figure below shows a normal distribution of the probability density function with $\mu=0$, $\text{var} = t$. The range of the parabola corresponds to variation of this normal distribution within one standard deviation.

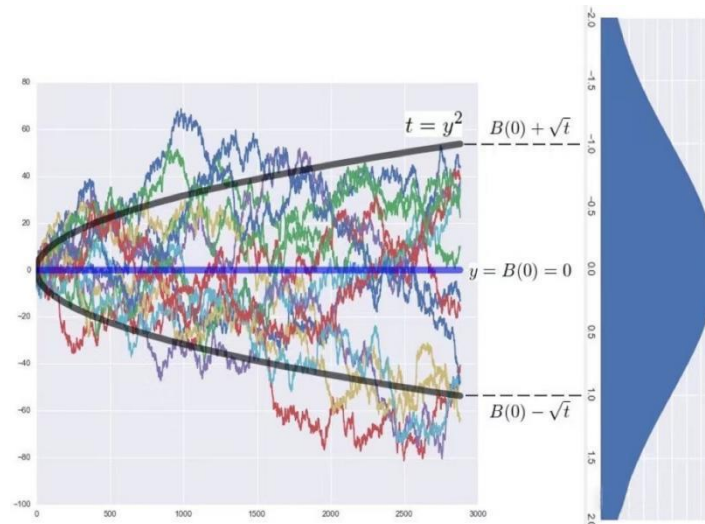


Figure 3. Distribution of the probability density function

These two characteristics suggest a strong possibility that the stock price will fluctuate between the initial price, as opposed to staying above or below it if we utilize Brownian motion to characterize the high-frequency intraday changes of the stock price. Furthermore, the stock price at time t won't deviated too far from the points where: beginning price $\pm \sqrt{t} \times$ standard deviation of price changes".

The probabilistic approach for calculating the Brownian motion extremes in time t is provided by the third property. Due to the fact that $B(t)$ has a mean of 0 and a variance of t , which follows that $\text{Prob}(M(t) \geq a) = 2\text{Prob}(B(t) \geq a)$. We can readily determine the likelihood $\text{Prob}(M(t) \geq a) = 2\text{Prob}(B(t) \geq a) = 2 - 2\Phi\left(\frac{a}{\sqrt{t}}\right)$ [1].

The standard normal distribution's cumulative distribution function is denoted by the symbol Φ . The Markovianity and Reflexivity of Brownian Motion may be used to demonstrate the following equation. Similar to that, if we let $m(t)$ be the Brownian motion in time 0 to t . The lowest value that may be attained is $m(t) = \min B(t) (0 \leq s \leq t)$, and using reflexivity again, one can calculate the probability that $B(t)$ will have a minimum value that is less than the specified threshold value as follows: $\text{Prob}(m(t) \leq -a) = \text{Prob}(M(t) \geq a) = 2 - 2\Phi\left(\frac{a}{\sqrt{t}}\right)$ [4].

The data above may quantify the probability distribution of the extreme value of the stock price if Brownian motion is employed to explain the stock price. This is helpful for managing risk and figuring out fair limit order prices when buying and selling stocks.

The final characteristic is essential to understanding Brownian motion as a stochastic process. It is continuous yet there is nowhere in the motion can be differentiated. The sample trajectories of Brownian motion above demonstrate its randomness which each of them keeps fluctuating up and down. The nowhere differentiable mean that the classical calculus was not working on Brownian motion [2]. This certainly paused the development of Brownian motion and led to the introduction of Itô Calculus.

4. Quadratic variation

Every continuous function $f(t)$ with quadratic variation defined as $\sum_{i=0}^{N-1} [f(t_{i+1}) - f(t_i)]^2$ for any time interval $[0, T]$ which divided by $\Pi = \{0 = t_0 < t_1 < t_2 < \dots < t_N = T\}$. Replace $f(t)$ in the function with the Brownian motion $B(t)$. According to a theory regarding the quadratic variation of $B(t)$, the quadratic variation of $B(t)$ equals T as the division of the time interval becomes smaller, $\max_i \{t_{i+1} - t_i\}$ tends to 0 [2]. Shown in the equation as follow:

$$\lim_{|\Pi| \rightarrow 0} \sum_i [B(t_{i+1}) - B(t_i)]^2 = T \text{ where } |\Pi| = \max_i \{t_{i+1} - t_i\}.$$

In brief, the quadratic variation of Brownian motion is T as a stochastic process. This is an essential property of Brownian motion, and the quadratic variation is of great importance in the derivation of the Itô integral. The fact that its quadratic variation is non-zero means that it fluctuates so often under the influence of randomness that regardless of how tiny the time interval, the total of the squares of the displacement differences on these intervals equals the length of the interval T .

5. Geometric Brownian Motion

Brownian motion with drift is the procedure $X(t) = ut + \sigma B(t)$ [4]. Process $X(t)$ tends to drift from the path at a rate of μ [2]. The normal distribution of $X(t)$ is $X(t) \sim N(\mu t, \sigma^2 t)$. As a linear combination of tiny changes in t and small increments of the Brownian motion $B(t)$, the minor changes in the process $X(t)$ may be represented as follows:

The slight changes in the process $X(t)$ may be represented as a linear combination of minute changes in t and tiny steps of the Brownian motion $B(t)$.

$$dX(t) = \mu dt + \sigma dB(t)$$

It is still not the greatest option to explain the stock price even with Brownian motion $X(t)$ with drift rate and variance parameter. The major justification for this is because, unlike stock prices, the values of $X(t)$ and $B(t)$ may both be negative. $X(t)$ may be used to express the rate of return since the stock return can be either positive or negative.

Assuming $S(t)$ is the stock price, $dS(t)$ represents the changes in the stock price over an infinitesimally small-time interval dt , and $dS(t)/S(t)$ represents the rate of return over this period, therefore $\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$.

The stochastic differential equation of $S(t)$ is $dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$, which is a geometric Brownian motion. There are several characteristics of geometric Brownian motion are useful for people to characterize prices:

1) Normal distribution: it has been empirically demonstrated that the successive compound returns of stock prices generally follows the normal distribution.

2) Brownian motion's characteristics allow us to infer that the stock price model discussed above is a Markov process, meaning the present price has all the information necessary to forecast the stock's future.

3) The nowhere differentiable and non-zero quadratic variation properties of Brownian motion are consistent with the existence of turning points in stock returns.

6. Itô calculus

The Brownian motion laid the foundation for the study of stock prices, but the nowhere differentiable property of Brownian motion slow impede the further development until the Itô calculus was proposed in 1944 by the Japanese mathematician Itô Kiyoshi opened the way to resolving the issue and established a strong basis for stochastic analysis [2].

The formula is similar to the chain rule from basic calculus. A differentiable function f of the real variable x . Consider the modifications $\Delta x = x - x_0$ and $\Delta f(x) = f(x) - f(x_0)$, assuming that x_0 is fixed. Calculus has shown that the following approximation of Taylor expansion is valid:

$$\Delta f(x) = f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O(\Delta x)^3$$

When x is infinitesimally near to x_0 , we may substitute the differential dx for Δx and get the following results:

$$df(x) = f'(x)dx + \frac{1}{2}f''(x)(dx)^2 + O(dx)^3$$

In the basic calculus, any phrases involving terms of the same or higher order than dx^2 are disregarded, and the following formula is attempted:

$$df(x) = f'(x)dx$$

The differential formulation of the chain rule may be obtained by inserting into the preceding formula and treating $x = x(t)$ as a differentiable function of t :

$$df(x(t)) = f'(x(t))dx(t) = f'(x(t))x'(t)dt [2].$$

A similar equation will be shown for the stochastic environment. In this instance, a stochastic process called B_t takes the role of the deterministic function $x(t)$. $F_t = f(B_t)$ represents the composition of the process B_t and the differentiable function f . We may suppose that the increase dB_t is overlooked in the same way that increments involving powers of dt^2 or above are. Then the expression becomes:

$$dF_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2$$

In the computation of dB_t we may take into the account quadratic variation that $dB_t^2 = dt$ or $dt dB_t = 0$. Therefore, we can have the general case of Itô formula:

$$dF_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

However, some relations like drift rate and diffusion parameter, in most situations, produce the increments dX_t . The increment provided by dX_t is a crucial circumstance:

$$dX_t = a(X(t), t)dt + b(X(t), t)dB(t)$$

By using Ito's Lemma and Let $f(X(t), t)$ to be a differentiable function of $X(t)$,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX)^2$$

Bring $dX = a(X(t), t)dt + b(X(t), t)dB$ into the function, omitting all the terms which has higher order of dt , the general formula of Ito's Lemma is finally obtained as:

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}a + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}b^2 \right)dt + \frac{\partial f}{\partial X}b dB$$

Ito's Lemma demonstrates that the Brownian motion in the dX expression is identical to the Brownian motion on the right side of the df expression. In other words, rather than two separate Brownian movements, the same Brownian motion determines the randomness in both f and X .

7. Black-Scholes Differential Equation

In this section, we provide an example of the BS differential equation for the European call option. The price of the European call option, symbolized by the symbol E , is a function of the underlying stock price S , and the time t shown as $E(S, t)$.

Using Ito's Lemma for E , we get:

$$dE = \left(\frac{\partial E}{\partial S}\mu S + \frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial S^2}\sigma^2 S^2 \right)dt + \frac{\partial E}{\partial S}\sigma S dB$$

The stochastic differential equations of S and E were then discretized to see how the stock price S and option price E changed over a short period of time Δt :

$$\Delta E = \left(\frac{\partial E}{\partial S}\mu S + \frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial E}{\partial S}\sigma S \Delta B$$

$$\Delta S = \mu S \Delta t + \sigma S \Delta B$$

If the function f of the Ito Process X is also Ito process, as was previously mentioned, and both stochastic processes' uncertainty is caused by the same Brownian motion. By virtue of this fact, the Brownian motion of the change in stock price and option price ΔE and ΔS enabling us to create a portfolio that eliminates the Brownian motion.

$$\begin{array}{l} \text{short 1 option} \\ \text{Long } \frac{\partial E}{\partial S} \text{ stock} \end{array}$$

This portfolio short 1 option and long $\partial E/\partial S$ shares of stock so that Brownian motion ΔB is perfectly eliminated. Delta hedging is the process of building a portfolio to get rid of stochasticity. Using P denotes the value of the portfolio, then the change in time interval Δt as:

Delta hedging is the process of building a portfolio to get rid of stochasticity. This portfolio fully hedges out Brownian motion by shorting one option and longing $\frac{\partial E}{\partial S}$ shares of stock. P represents the portfolio's value, followed by the change in time interval Δt as:

$$\Delta P = \left(-\frac{\partial E}{\partial t} - \frac{1}{2} \frac{\partial^2 E}{\partial S^2} \sigma^2 S^2\right) \Delta t$$

By selling 1 option and simultaneously buying $\partial E/\partial S$ shares, we construct a risk-free portfolio. Since no market allows for risk-free arbitrage, this portfolio's rate of return is equal to the risk-free rate of return, or $\Delta P = rP\Delta t$. By combining ΔP and $P = -E + \left(\frac{\partial E}{\partial S}\right) S$ in the equation above, it was determined that:

$$\frac{\partial E}{\partial t} + rS \frac{\partial E}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 E}{\partial S^2} = rE$$

The differential equation above is known as Black-Scholes equation, which is differential equation rather than stochastic differential equation since there are no random variables in this equation due to the elimination of uncertainty via delta hedging.

8. Limitations and Conclusion

Despite significant failures like the financial crisis of 2008–2009, which was linked to the improper use of trading models, quantitative model-based trading is still on the rising trend. Since no model is flawless, understanding its flaws may help traders make wise judgments and prevent pricey errors that might lead to significant losses [6].

The following are some typical Black-Scholes model restrictions:

1) Assumes that volatility and the risk-free rate of return will stay constant throughout the term of the option. None of those ideas will necessarily hold true in the real world [8].

2) Assumes that trading will always be free of charge, excluding the consequences of liquidity risk and brokerage charges.

3) Ignores the more frequent price changes and assumes that stock price movements will follow a lognormal pattern, which follows random walk pattern or geometric Brownian motion pattern [9].

4) Assumes there is no provision for dividend payout, eliminating their impact on value changes.

Even though the Black-Scholes formula is widely acknowledged, we cannot blindly follow any mathematical model because doing so could result in uncontrolled risk exposure. For instance, the improper application of trading models led to the financial crisis. Model utilization is still constantly changing and remains the main foundation for trading despite the challenges. So, if you want to use the model carefully, you need to know all about its limitations, their effects, and any possible alternatives to safe trading [10].

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