Planimeter: A Magical Tool to Calculate Area

Yuteng Wang

The Hong Kong Polytechnic University, Hong Kong, 999077, China

Abstract. It might sound unbelievable that a specific combination of rods and wheels can be used to calculate the area of any closed curve, especially after hearing that it was invented in the 19th century. The magical tool is called planimeter and it achieves high accuracy in measuring an area by merely tracing around its boundary once using the tracer. The planimeter's advent has brought immeasurable convenience in surveying and measuring. This article introduces the main versions of it, including its history, mechanics, and the working principle.

Keywords: A Specific Combination of Rods and Wheels; Area of Closed Curve; Planimeter.

1. Development

People have long come up with equations for calculating the area of basic geometries such as rectangles, triangles, and circles. These equations not only help increase calculation speed, but also, to some extent, simplify complicated geometries by subdividing them into the basic ones.

However, when calculating over-curvy geometries, such as the area of a parabola enclosed by the x-axis, or the area of a randomly drawn black shade that does not even have an expression, these techniques fell short. Though primitive methods still work in this case, the time-consuming and error-prone feature prohibit their application. Later, the invention of calculus greatly improves the speed and accuracy of such computation. However, some functions may be difficult to integrate, and some geometry may be too edgy and twisted (like the contour of a lake) that even constructing a function seems impossible. A faster approach is still needed by scientists.

Human, again, showed their genius in solving this problem by inventing the planimeter. The birth of the planimeter can be traced back to 1814 when Johann Martin Hermann made the cone planimeter to calculate the area based on integration [1]. The original drawing of this magical tool is shown in Figure 1, this machine ingeniously converts the area into the position and movement of the pin. As shown in Figure 2. When it functions, the pin will change both its x and y coordinates. Changing the y-coordinate will change the position of the gear on the cone, which changes its rotational speed. Changing the x-coordinate will rotate the cone, therefore rotate the pointer above.

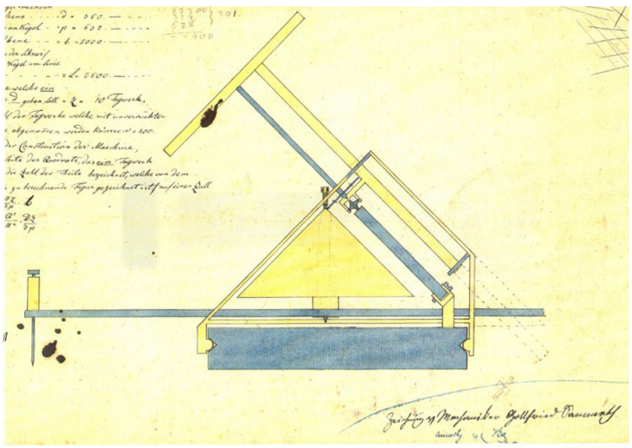

Figure 1. The original drawing of Hermann. https://www.researchgate.net/figure/Original-diagram-of-Hermanns-planimeter-Courtesy-of-Professor-Fischer-Muenchen_fig1_257930030

The cone planimeter was a monumental achievement, however, it pales in comparison with Jacob Amsler's invention: the polar planimeter [2]. Jacob designed it in 1854, which is about 170 years ago.

However, both the simplicity regarding its operation and its size, and the accuracy of measurement still makes it a significant influence on contemporary scientists.

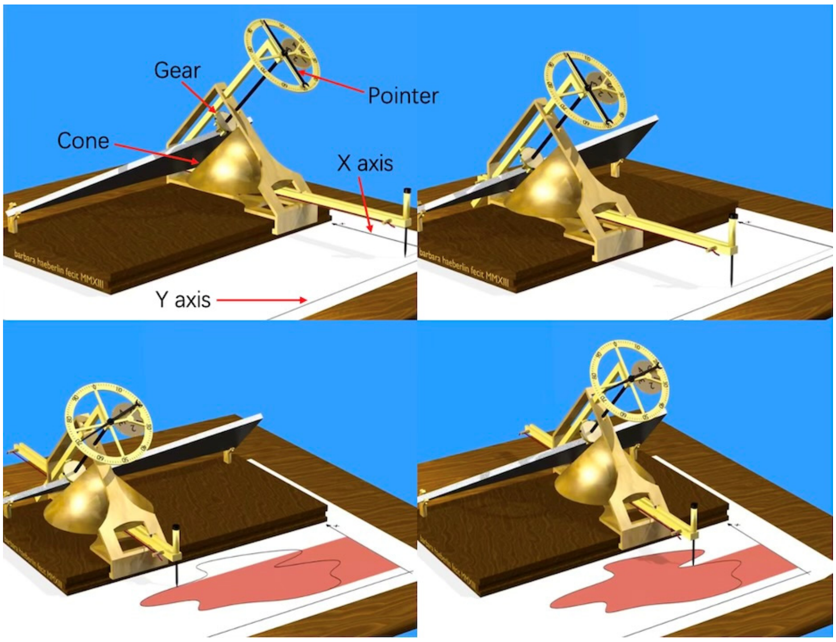

Figure 2. The working mechanism of the cone planimeter. https://www. youtube.com/ watch?v=_ W35iDhRfZg&t=3s

Figure 3 shows a typical planimeter with its structure being highlighted, we will now elaborate each part in detail. The tracer is used to trace the curve, and the pole is used as a nail to fix the position of the pole arm such that it could only rotate around the pole. The pivot's position and the angle between the tracer arm and the pole arm change flexibly while drawing, the measuring wheel and dial can show the area of the region bounded by the curve. When the arm is moving, the wheel also rotates. However, the rotational speed only equals in magnitude to the projection of the velocity on the line perpendicular to the tracing arm. Therefore, the wheel will partially rotate and partially slide when tracing. This device was proven highly accurate using modern technologies as a reference, with an average measuring error of around 2%.

Aside from the polar planimeter, there is another version: the linear planimeter. As shown in Figure 4, the working principle is similar to a polar planimeter, except that the pivot was attached to the middle of a rod that can only slide horizontally with the position of the tracer T. This type of planimeter is typically used to measure the area of narrow and long geometries.

In 1875, Danish mathematician Holger Prytz designed the "hatcher" planimeter [3], where only one rigid component was needed (Figure 5), however, the accuracy decreased relative to the previous two.

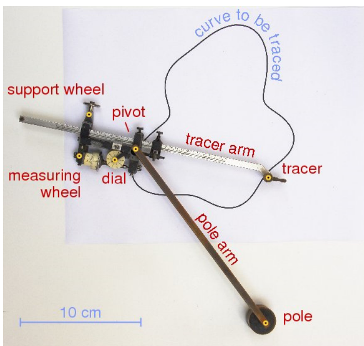

Figure 3. The polar planimeter. http://www.ams.org/publicoutreach/feature-column/fcarc-surveying-two

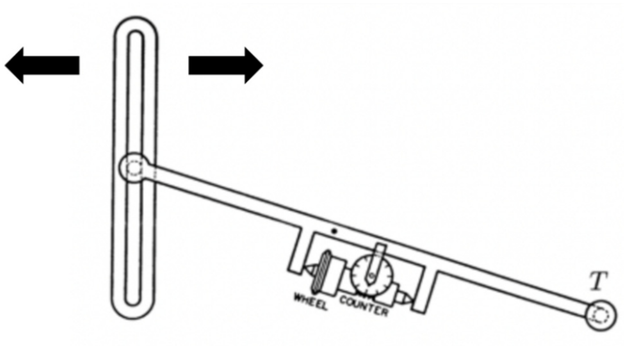

Figure 4. The linear planimeter. http://persweb. wabash.edu/ facstaff/ footer/ Planimeter/ Polar& Linear.htm

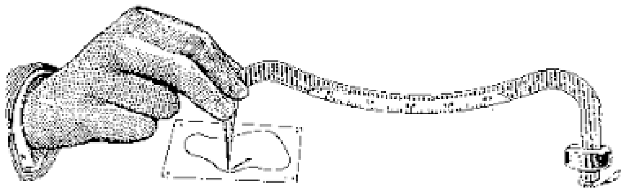

Figure 5. The Prytz planimeter. http://persweb. wabash.edu/ facstaff/ footer/ Planimeter/ Prytz/ PrytzArea.htm

Although these planimeters are somewhat obsolete in the face of modern technologies, it is still fascinating to study the intricate principle behind them and to understand the intelligence of these people from more than a century ago.

2. Working Principle

This section will only cover the mechanism behind the polar planimeter, and a simplified mathematical description is used for an idealized analysis.

To understand the principle, one can simplify the device into the "two rods, one wheel" system. Figure 6 is the simplified version we will use for the rest of this section, in which the red line denotes the wheel, and the rest are self-explanatory. For convenience of discussion, the wheel is put between the pivot and the tracer, and the dial is neglected. Therefore, it is necessary to assume that the wheel is giant enough to not roll a full circle while tracing. There will be numbers marked around the wheel such that the number 0 is at the bottom of the wheel when the tracing starts. After a whole trace around the curve, the new number at the bottom denotes the area bounded by this closed curve. Why the number can accurately show the area regardless of the shape of the curve, and whether the net roll of the wheel is proportional to the area will be discussed in the following content.

Before we formulate the mathematical description of the planimeter, three restraints are to be addressed:

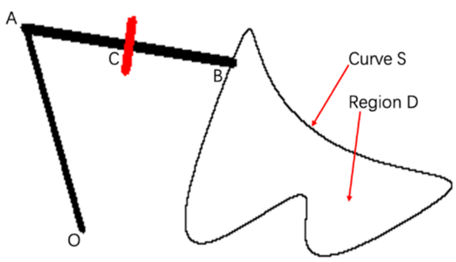

Figure 6. The simplified version of planimeter. http://persweb. wabash.edu/ facstaff/ footer/ Planimeter/ Polar&Linear.htm

1. The distance between O and any point on S needs to be bigger than |OA-AB|, and smaller than OA+AB so that every point on S can be traced.

2. S should be piecewise smooth, simple, and closed, where simple means not crossing itself, if it's not simple, it can be separated into smaller simple ones.

3. O is normally put outside the curve.

The wheel is assumed to be placed at B first (which means C coincides the B, even though it is practically impossible) which makes mathematical processing easier for demonstration and understanding. Derived from the first constraint above, the pink ring in Figure 7, consisting of two concentric circles with radius |OA-AB| and OA+AB, will be the region the tracer can reach (boundary excluded). For an arbitrary tracing point B in this region, there exist two pivot points A_1 and A_2 where the pivot can lie upon, and they are symmetrical about the line OB. Therefore, theoretically the planimeter can have two distinct positions when tracing to the point B. However, in practice it's restricted to only one position since for a planimeter's pivot to change from A_1 to A_2 through merely tracing, it must be completely extended, which means the tracing point B must touch the boundary of the big circle, thus it cannot happen because the region is boundary-excluded (the same for the small inner circle). Therefore, at any point, there will always be a unique position of the planimeter.

Assume OA_1B is the unique position, then we can draw a unit vector $\boldsymbol{n}$ normal to A_1B at point B. As a result, every point in the pink region corresponds to a unique normal vector $\boldsymbol{n}$, therefore, all normal unit vectors will form a vector field [4] in the ring (see Figure 8).

The meaning of creating such a field is to calculate the net roll of the wheel by turning it into a line integral around the curve S. We assume that at one point on S, the unit tangent vector of S is $\boldsymbol{T}$, and the tracer velocity vector is $\boldsymbol{V}$ of which the same direction as $\boldsymbol{T}$. If we ignore the change in $\boldsymbol{V}$ during an extremely short time dt, the tracer will travel the distance $ds = |\boldsymbol{V}|dt$ during this short interval. Since the speed of the wheel is quantitively equivalent to the projection of the tracer velocity $\boldsymbol{V}$ on $\boldsymbol{n}$, it is apparent that the wheel will rotate the length of $dt\langle \boldsymbol{n}, \boldsymbol{V}\rangle = ds\langle \boldsymbol{n}, \boldsymbol{T}\rangle$. After a whole trace, the net roll of the wheel will be $\oint_S ds\langle \boldsymbol{n}, \boldsymbol{T}\rangle$.

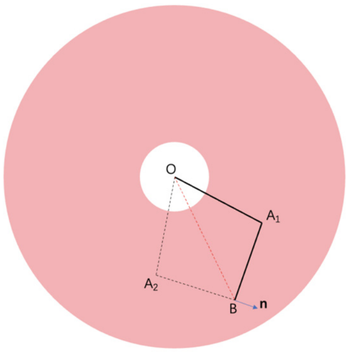

Figure 7. Two possible positions of the planimeter

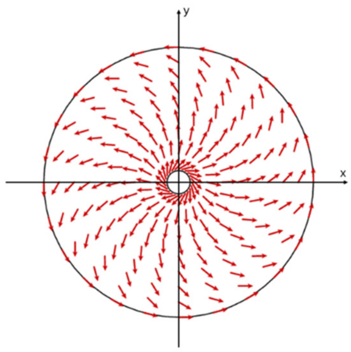

Figure 8. The vector field of all the $\boldsymbol{n}$'s. http://www.ams.org/publicoutreach/feature-column/fcarc-surveying-two

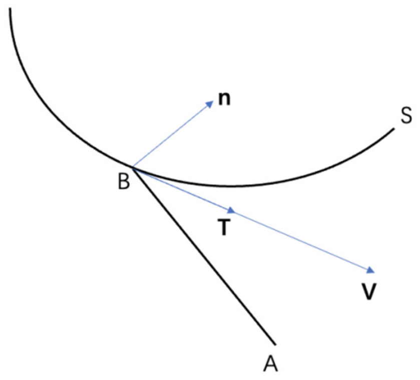

Figure 9. The relationship between $\boldsymbol{n}$, $\boldsymbol{T}$, $\boldsymbol{V}$, and ds at one point on S

To find out the relationship between $\oint_S ds\langle \boldsymbol{n}, \boldsymbol{T}\rangle$ and the area of region D, it's necessary to find the connection between the line integral and the double integral. Therefore, the Green's Theorem in the following form is applied:

$$\oint_S \langle \boldsymbol{n}, \boldsymbol{T}\rangle * ds = \oint_S \langle \boldsymbol{n}, d\boldsymbol{r}\rangle = \oint_S p(x,y) * dx + q(x,y) * dy = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) * dx * dy$$

Here's how this equation works: As shown in Figure 8, for every $\boldsymbol{n}$ in the pink region, there exist two scalars $p(x,y)$ and $q(x,y)$, such that $\boldsymbol{n} = p(x,y)\boldsymbol{i} + q(x,y)\boldsymbol{j}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in *x* and *y* direction accordingly. Meanwhile the tangent vector of curve S with length ds can be expressed as $d\boldsymbol{r} = (ds)\boldsymbol{T} = (dx)\boldsymbol{i} + (dy)\boldsymbol{j}$. Therefore, the first and second equality in the equation apparently holds, and the third quality holds as well since it itself is the Green's theorem. Once the expression of $p(x,y)$ and $q(x,y)$ is found in terms of *x* and *y*, we are then able to compute the right side of the partial differential part. The result is proportional to the area of D if it equals a nonzero constant.

By looking at the vector field it might seem daunting to calculate $p(x,y)$ and $q(x,y)$. However, one can observe that all the vectors lying at the same distance to the pole have the same magnitude and direction of angle rotated away from its diameter. Therefore, it's easier to solve the polar form of the vectors, then convert it into cartesian form.

Every vector $\boldsymbol{n}$ has a length 1. The angle, as previously mentioned, can be expressed as $\theta + f(r)$, where θ is the angle between OB and *x*-axis, and $f(r)$ is the offset angle dependent of the distance to the pole. Thus, the polar form of this vector is $(1, \theta + f(r))$, therefore, $p(x,y)$ and $q(x,y)$ satisfy:

$$\begin{cases} p^2 + q^2 = 1 \\ \tan^{-1}\dfrac{q}{p} = \theta + f(r) \end{cases}$$

Solving it yields:

$$\begin{cases} p = \cos\left(\tan^{-1}\left(\dfrac{y}{x}\right) + f\left(\sqrt{x^2+y^2}\right)\right) \\ q = \sin\left(\tan^{-1}\left(\dfrac{y}{x}\right) + f\left(\sqrt{x^2+y^2}\right)\right) \end{cases}$$

We can the calculate $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$:

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = \frac{\partial}{\partial x}\left\{\sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\cos\left(f\left(\sqrt{x^2+y^2}\right)\right) + \cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\sin\left(f\left(\sqrt{x^2+y^2}\right)\right)\right\}$$
$$-\frac{\partial}{\partial y}\left\{\cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\cos\left(f\left(\sqrt{x^2+y^2}\right)\right) - \sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\sin\left(f\left(\sqrt{x^2+y^2}\right)\right)\right\}$$
$$= \frac{\partial}{\partial x}\left\{\frac{y}{\sqrt{x^2+y^2}}\cos\left(f\left(\sqrt{x^2+y^2}\right)\right)\right\} - \frac{\partial}{\partial y}\left\{\frac{x}{\sqrt{x^2+y^2}}\cos\left(f\left(\sqrt{x^2+y^2}\right)\right)\right\}$$

$$+\frac{\partial}{\partial x}\left\{\frac{x}{\sqrt{x^2+y^2}}\sin(f\,(\sqrt{x^2+y^2}))\right\}+\frac{\partial}{\partial y}\left\{\frac{y}{\sqrt{x^2+y^2}}\sin(f\,(\sqrt{x^2+y^2}))\right\}$$

We will now show that the two partial differential equations in the third line are inherently the same. Write the first equation as $g(x,y)$, and the second as $g(y,x)$, then the difference the two equations are zero for every point (x_0,y_0) in the region:

$$\frac{\partial g(x,y)}{\partial x}-\frac{\partial g(y,x)}{\partial y}\Big|_{(x_0,y_0)}=\lim_{h\to 0}\left(\frac{g(x_0+h,y_0)-g(x_0,y_0)}{h}\right)-\lim_{m\to 0}\left(\frac{g(x_0+h,y_0)-g(x_0,y_0)}{m}\right)=0$$

Before computing the remaining components, $f(\sqrt{x^2+y^2})$ needs to be known. We will use $f(r)$ for convenience.

Figure 10 shows one of the placements of the planimeter. $f(r)$ denotes the angle between n and OB away from O, and r equals to |OB|. If we denote |OA|$=a$, |AB|$=b$, so by law of cosines, $\cos\angle ABO=\frac{b^2+r^2-a^2}{2br}$, since $f(r)+\angle ABO=\frac{\pi}{2}$, so $f(r)=\sin^{-1}\frac{b^2+r^2-a^2}{2br}$.

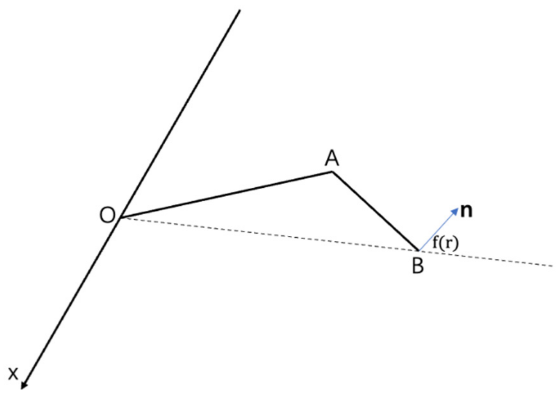

Figure 10. The solution of $f(r)$

Plug $f(r)$ into the equation:

$$\frac{\partial}{\partial x}\left\{\frac{x}{\sqrt{x^2+y^2}}\sin(f\,(\sqrt{x^2+y^2}))\right\}+\frac{\partial}{\partial y}\left\{\frac{y}{\sqrt{x^2+y^2}}\sin(f\,(\sqrt{x^2+y^2}))\right\}$$
$$=\frac{\partial}{\partial x}\left\{\frac{x(b^2+x^2+y^2-a^2)}{2b(x^2+y^2)}\right\}+\frac{\partial}{\partial y}\left\{\frac{y(b^2+x^2+y^2-a^2)}{2b(x^2+y^2)}\right\}$$
$$=\frac{2b(x^2+y^2)(2b^2+4x^2+4y^2-2a^2)-4b(x^2+y^2)(b^2+x^2+y^2-a^2)}{4b^2(x^2+y^2)^2}=\frac{2(x^2+y^2)^2}{2b(x^2+y^2)^2}=\frac{1}{b}$$

Therefore, we can conclude that $\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}$ is a constant, therefore $\oint_S p(x,y)dx+q(x,y)dy=\iint_D\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dxdy=\iint_D\frac{1}{b}dxdy=\frac{1}{b}(Area)$, it can be observed that the area depends on the length of the second rod, in this case, it is b. However, in reality, what the area depends on is the distance between the pivot and the wheel. It is we assigned the wheel to coincide with the tracer that makes the distance equal b. To prove this, assume that the wheel is between the pivot and the tracer, and cut the second rod of the planimeter shorter, such that the new end meets where the wheel sits, we then assign a new tracer to the end. Although the vector fields will be changed, all the relations in the derivation process will remain the same and no rules or premises will be violated, which proves that the distance matters instead of the length.

3. Limitation

Despite the rigorous process we went through, there still exists some ambiguity to be cleared. All the above process only addresses the condition where the wheel is moving without changing the direction. Our discussion assumes that the infinite ds is the only part where the wheel is moving (see Figure 13). However, according to John Eggers [5], between every two ds, the wheel should make an extremely small turn to change the direction of rolling, and that rotation itself adds to the net roll of the wheel. Consider fixing one end of a rod and the other end can freely rotate, the wheel at the other end is turning as long as the rod is rotating, but the wheel has a net roll.

However, we don't need to consider this case if we place the pole outside the curve, in this case, the net rotation angle after one trace will be canceled out.

4. Conclusion

This article covers the history of the planimeter, which provides an insight into how people before the information age can measure areas easily and accurately, as well as how math equations and theorems are integrated into engineering. The working principle behind one of the planimeters covered provides a basic math principle of all the planimeters even though the rest were not addressed here. By understanding these machines, we can not only understand the power of math in daily life, but also study and admire the wisdom of the people who built such intricate tools.

References

[1] Foote, R. L. (2009). Pre-Amsler Planimeters. Http://persweb. wabash.edu/ facstaff/ footer/ Planimeter/ PreAmsler.htm.

[2] Foote, R. L. (2009). How planimeters work. Http://persweb. wabash.edu/ facstaff/footer/ Planimeter/ HowPlanimetersWork.htm.

[3] Foote, R. L. (2009). Prytz Planimeters. http://persweb. wabash.edu/ facstaff/ footer/ Planimeter/ Prytz/ Prytz.htm.

[4] Casselman, B., Eggers, J. (2008). The Mathematics of Surveying: Part II. The Planimeter. American Mathematical Society Feature Column, June-July. http://www.ams.org/publicoutreach/feature-column/ fcarc-surveying-two.

[5] Eggers, J. (2020). A Closer Look at the Compensating Polar Planimeter. The College Mathematics Journal, 51:2, 105-116, DOI: 10.1080/07468342.2020.1702852.