The Principle of Variational Method and The Use of Variational Method to Solve Practical Problems

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Abstract. This article tries to obtain and use the fundamental principle of analytical mechanics—variational method, which is practical application of calculus. With the development of mathematic and physics, more people want to use advanced mathematics to solve practical problems. Hence, variational method has been developed. The article analyzed two circumstance after obtaining the Euler—Lagrange equation and the first integral. Obtaining the most rapid descent line is also an application of mathematical modeling, which is a classical application of mathematic. The article will focus on two classical circumstance, using the variable methods, and the basic logic of the methods.

Keywords: Applied mathematics, Variation, Calculus, Newton, brachistochrone.

1. Introduction

As for the origin of the method, there's a story. In 1696, Johann Bernoulli set a challenge to global mathematicians *what shape should an object glide down a smooth, frictionless trajectory from a standstill, if the descent is required to take the shortest time,* which is the famous brachistochrone problem. The problem is so interesting that it has attracted the attention of many mathematicians. Johann himself came up with a solution using optical analogies, and his more impressive brother Jacob Bernoulli came up with another solution.[1] In addition, the famous Gottfried Leibniz, Guillaume de L'Hopital and others have given their own solutions. Word soon reached Issac Newton, an established scientist. One day in January 1679, the Lord, who was now serving at the Royal Mint, returned home from work to find a challenge from Johann Bernoulli. [2] Amazingly, it only took the Lord a night to find the solution. The Lord quietly sent his answer back anonymously. His solution, however, was so dashing and impressive that Johann soon realized who the real author was, even without the credit. In response, he famously remarked: I recognize the lion by his claw mark.

Variational method is differentiation on infinite dimensional space, researchers generally call it Frechet differentiation, is actually a generalization of differentiation in infinite dimensional space. A Frechet differential acting on a functional is called a variation.[3] There's an old way of saying that the independent variable of the function space is called the scalar, and when the scalar changes a little bit, this is actually a variation. Variation is the extension of differentiation in function space, and its spiritual connection is the same. The methods can be used in solving functional variational method, dynamic programming and optimal control.

Variational method is the field of mathematics dealing with functional functions, as opposed to traditional calculus dealing with functions.[4] The problem of finding an extreme value of a functional is called a variational problem, and the function that takes an extreme value of a functional is called a solution of a variational problem, also called an extreme value function.[5] It is a traditional but significant method to find out the maximum or minimum of an integral, with the assistance of Euler—Lagrange equation. Plenty of problems in reality can be solved by variational method, for example, how to find out the fastest trajectory among all the locus of a falling ball from a fixed-point A to another fixed point. [6]

2. Methods and Theory

2.1. Background knowledge and method

Suppose there're two settled points *(a,p)* and *(b,q)*, the equation *y=y(x)* of any curve connection these two points will have the conditions.

$$y(a) = p, y(b) = q \tag{1}$$

Now consider the definite integral of the following form.

$$I = \int_a^b f(y, y')dx \tag{2}$$

Where *f(y,y')* is a function of *y(x)* and the first derivative *y'(x)*, the author expect to find a concrete *y(x)* such that *I* has extremum. Applying similar theory in calculus, the author can know that: when *I* is taken to the extreme value, *y(x)* is changed slightly, and I should remain the same under the first-order approximation, as shown in Figure 1.

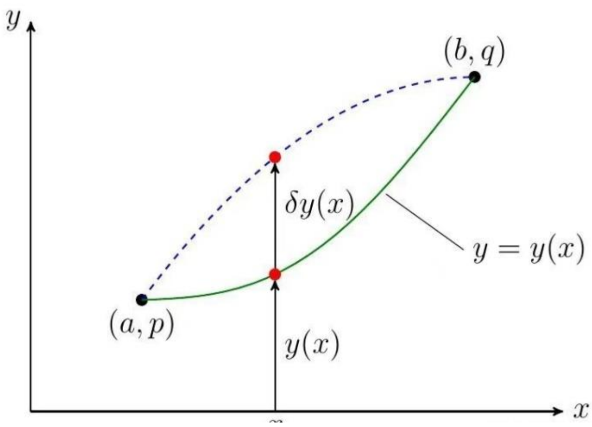

Figure 1 The Euler—Lagrange equation

If *y(x)* has a small change $\delta y(x)$, then the change of *f(y,y')* is

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \tag{3}$$

The change of *I* is

$$\delta I = \int_a^b \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right] dx \tag{4}$$

Then using integrating by parts, the results can be written as

$$\int_a^b \frac{\partial f}{\partial y'}\delta y' dx = \int_a^b \frac{\partial f}{\partial y'} d(\delta y) = \left[\frac{\partial f}{\partial y'}\delta y\right]_a^b - \int_a^b \delta y \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx \tag{5}$$

Since the boundary conditions for y(x) are fixed, $\delta y(a) = \delta y(b) = 0$, the first term resulting from integration by parts is 0, with only the second term contributing. Substitute (5) back to (4), the author will get after a little simplification.

$$\delta I = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y(x) dx \tag{6}$$

If I have an extreme value, for any $\delta y(x)$ satisfying the boundary condition, there must be $\delta I = 0$, which requires

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \tag{7}$$

This is the Euler—Lagrange equation, which will be widely used in later article.[7] Next the author will derive a famous conclusion of Euler's Lagrange formula using another method to derivate the Euler—Lagrange equation. By definition of derivative

$$\delta F=F[y+\delta y]-F[y]=\int_a^b f(x,y+\delta y,y'+\delta y')dx-\int_a^b f(x,y,y')dx \tag{8}$$

Reserve one order for the above results using Taylor expansion

$$\delta F=\int_a^b\left[f(x,y,y')+\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial y'}\delta y'\right]dx-\int_a^b f(x,y,y')\,dx \tag{9}$$

Then, calculating the above integral, result in (9) can be written as

$$\delta F=\left[\frac{\partial f}{\partial y'}\delta y\right]_a^b+\int_a^b\left[\frac{\partial f}{\partial y}\delta y-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\right]dx=\int_a^b\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y dx \tag{10}$$

Because the function is fixed at a and b, $\delta y(a)=0,\delta y(b)=0$. If the variation δF is to obtain an extreme value, it must be satisfied that

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \tag{11}$$

The complete derivative off is

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+y''\frac{\partial f}{\partial y'} \tag{12}$$

The second term can be obtained by substitution the Euler—Lagrange formula, then result became to

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+y''\frac{\partial f}{\partial y}=\frac{\partial f}{\partial x}+\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) \tag{13}$$

Hence, the equation has been got

$$\frac{d}{dx}\left(f-\frac{dy}{dx}\frac{\partial f}{\partial(\frac{dy}{dx})}\right)=\frac{\partial f}{\partial x} \tag{14}$$

Which is the famous first integral of the Euler—Lagrange formula.

2.2. Structure and content of the article

In the Sec. 3.1, the article focuses on the shortest path between the two points using the variational method and obtains the result. For the simplest circumstance, the answer is a straight line. Then, in the Sec. 3.2, the article focuses on the classical problem --what shape should an object glide down a smooth, frictionless trajectory from a standstill, if the descent is required to take the shortest time? By using the first integral and the Euler—Lagrange equation, an expression has been obtained. Then comparing the result with the standard formula of cycloid, the article proved the most rapidly decreasing curve is a cycloid.[7]

3. Results and Application

3.1. The shortest path between two points

If there're 2 points (a,p) and (b,q) , the curve with the shortest length connecting them will be found out later.The length of the element between two similar points (x,y) and (x+dx,y+dy) on the curve is

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{15}$$

The length of the whole curve is

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{16}$$

Now that the author wants S to have a minimum, thus the author take $f\left(y, \frac{dy}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, and use the Euler—Lagrange equation to find a function y(x) that minimizes S

$$\frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial y'} = \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \tag{17}$$

Taking the results back to (7), the author will get

$$\frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right) = 0 \tag{18}$$

The derivative of the results in the parenthesis is 0, so the results must be const. Hence the author will find that y' must be satisfied the equation of a line.[8]

$$y = kx + c \tag{19}$$

And because of the origin and the final are both given, the line equation can be found out easily.

3.2. The most rapid decline curve

The question has been described at the beginning, thus later the article will solve the problem directly, as shown in Figure 2.

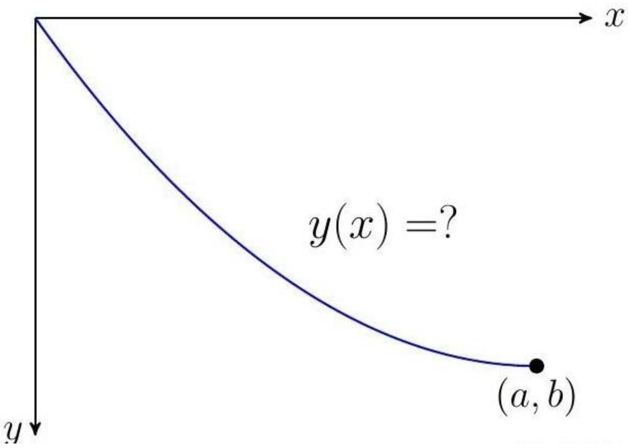

Figure 2 The rapid decline curve

When the object slides to the (x,y) position, its velocity can be solved based on the conservation of energy.

$$\frac{1}{2}mv^2 = mgy, v = \sqrt{2gy} \tag{20}$$

By definition, velocity is equal to the length of the track traveled per unit time.

$$v = \frac{ds}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\frac{dx}{dt} \tag{21}$$

Combining (20) and (21), the author can get

$$dt = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2gy}}dx \tag{22}$$

After integrating, the author will get the expression of the total time

$$T = \frac{1}{\sqrt{2g}}\int_0^a \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{y}}dx \tag{23}$$

To find a minimal y(x) for T, the article can take $f(y, \dot{y}) = \sqrt{\frac{1+(\dot{y})^2}{y}}$ and apply the Euler—Lagrange equation

$$\frac{\partial f}{\partial y} = -\frac{1}{2}\sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{y^3}}, \frac{\partial f}{\partial \dot{y}} = \frac{\frac{dy}{dx}}{\sqrt{y(1 + (\frac{dy}{dx})^2)}}$$

Substitute the results back to (7), the results can be changed into

$$\frac{1}{2}\sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{y^3}} + \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{y\left(1 + \left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \tag{24}$$

Then, by calculation, the results can be shown as

$$\frac{1}{2}\sqrt{\frac{1+(\dot{y})^2}{y^3}}+\frac{\ddot{y}}{\sqrt{y(1+(\dot{y})^2)}}-\frac{1}{2}\frac{(\dot{y})^2}{\sqrt{y^3(1+(\dot{y})^2)}}-\frac{(\dot{y})^2\ddot{y}}{\sqrt{y(1+(\dot{y})^2)^3}}=0$$

$$\frac{1}{2}\left(1+\left(\frac{dy}{dx}\right)^2\right)^2+y\left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d^2y}{dx^2}-\frac{1}{2}\left(\frac{dy}{dx}\right)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)-y\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}=0 \qquad (25)$$

Thus, the results have been worked out

$$\frac{1}{2}\left(1+\left(\frac{dy}{dx}\right)^2\right)+y\frac{d^2y}{dx^2}=0 \qquad (26)$$

Then, the author only needs to solve the second ODE. Because the equation is non—liner ones, the author multiple 2y’ on both sides.

$$\frac{dy}{dx}\left(1+\left(\frac{dy}{dx}\right)^2\right)+2y\frac{dy}{dx}\frac{d^2y}{dx^2}=0 \qquad (27)$$

By observing (27), the author finds that the results above is a total differential. Thus, the results can be written as

$$\frac{d}{dx}\left[y\left(1+\left(\frac{dy}{dx}\right)^2\right)\right]=0 \qquad (28)$$

It is easy to know, that only if the results are const, the derivative of it can be 0

$$y\left(1+\left(\frac{dy}{dx}\right)^2\right)=c$$

$$(\frac{dy}{dx})^2=\frac{c}{y}-1=\frac{c-y}{y}$$

$$\dot{y}=\sqrt{\frac{c-y}{y}} \qquad (29)$$

Then, by separating variables, the author will get

$$x=\int dx=\int\sqrt{\frac{y}{c-y}}dy \qquad (30)$$

Using substitution, setting $y=k(\sin t)^2$, then x can be expressed as

$$x=\int\sqrt{\frac{c(\sin t)^2}{c-c(\sin t)^2}}d(c(\sin t)^2)$$

$$x=\int\frac{\sin t}{\cos t}2c\sin t\cos tdt$$

$$x=\int 2c(\sin t)^2dt=c\int(1-\cos 2t)dt$$

$$x=ct-\frac{1}{2}c\sin 2t+C \qquad (31)$$

Next, the article will take y back to the result

$$x(y)=c\sin^{-1}\sqrt{\frac{y}{c}}-\sqrt{y(c-y)}+C \qquad (32)$$

Because the origin of the trajectory is (0,0), C=0. Because the final of the trajectory is(a,b), there must have

$$c\sin^{-1}\sqrt{\frac{b}{c}}-\sqrt{b(c-b)}=a \tag{33}$$

Y(x) can be written as parameter equation, otherwise people will not find it is a cycloid

$$x(m)=R(m-\sin m), y(m)=R(1-\cos m) \tag{34}$$

Where R=0.5k, which is the radius of the process.

$$\frac{dy}{dx}=\frac{\frac{dy}{dm}}{\frac{dx}{dm}}=\frac{\sin m}{1-\cos m}=\frac{\sqrt{1-(\cos m)^2}}{1-\cos m}=\sqrt{\frac{1+\cos m}{1-\cos m}} \tag{35}$$

$$\sqrt{\frac{c-y}{y}}=\sqrt{\frac{c-\frac{c}{2}(1-\cos m)}{\frac{1}{2}c(1-\cos m)}}=\sqrt{\frac{1+\cos m}{1-\cos m}} \tag{36}$$

By checking computations, (35) and (36) are totally same. Thus, brachistochrone is a cycloid.

4. Summary

Variational method is an important part of analytical mechanics, with this tool, dealing with some complex physical models will be handy. In applied mathematics, dealing with extreme values of a function is equally important. The famous functional analysis is based on this theory. Operators, mappings, functionals and Hilbert operator spectrum theory are all established on it. In functional analysis, although analytical mechanics problems are too easy to study, this idea—after proving that they are continuous and convergent, the less well-behaved mathematical elements are put into function, is still important.

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