Comparing and Contrast Different Numerical Approaching to the IVP and BVP Problems

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Abstract. This paper focuses on the comparison of various numerical solutions for Initial Value Problem(IVP) and Boundary Value Problem(BVP) in differential equations. The practical problem is solved by comparing each numerical method solution. In the part of IVP problem, this paper established an Susceptible Infectious Recovered Model (SIR) to measure the disease situation in India during the new coronavirus period of 2020-2021, and calculated the corresponding parameters through machine learning genetic algorithm(GA). The differential equations are solved by using first and second-order Taylor expansions(TM), Euler(EM), improved Euler(ME), Runge-Kutta 2(RK2), optional Runge-Kutta(RK2*), and Runge-Kutta 4(RK4) methods. Then, based on the model, the situation of the novel coronavirus epidemic in the next year is predicted; In the part of BVP problem, this paper establishes a linear spring model(LSM) to measure the displacement of elastomer under the force distribution, that is, the force density, by using a simulation algorithm(SA) to expand a small amount of measurement data, and by using machine learning algorithm linear regression(LR) to calculate the spring stiffness coefficient. Numerical solutions are calculated by using forward, center and backward finite difference method(FDM), Galerkin method in variational method(VGLM), collocation method(CM) and Galerkin method in finite element method(FEM), and mean square error(MSE) is used to evaluate the model. Finally, the analytical solutions especially Laplace Transformation(LT) of the two models are given to compare the numerical solutions.

Keywords: Numerical Approching, Genetic Algorithm, Maching Learning, SIR model, Linear Regression, LSM model, Anylatical Approching, Laplace Transformation, MSE.

1. Introduction

This paper focuses on solving initial value problems (IVPs) and boundary value problems (BVPs) for ordinary differential equations (ODEs)[5], with an emphasis on obtaining numerical solutions[2][4]. In the field of applied mathematics, ordinary differential equations(ODEs)[6] play a crucial role in modeling various physical, biological, and engineering systems. Solving these equations often requires numerical methods, especially when analytic solutions are difficult or impossible to obtain. This paper focuses on the comparison of different numerical solutions for initial value problems (IVP) and boundary value problems (BVP) associated with differential equations.

For the IVP section, a Susceptible-Infected-Recovered (SIR) model is developed to assess the COVID-19 pandemic situation in India during 2020-2021[12][13][26]. The model parameters are calculated using machine learning techniques, specifically genetic algorithms (GA)[14]. Various numerical methods, including first-order and second-order Taylor expansions(TM)[1], Euler's method(EM)[19], modified Euler's method(ME)[20], Runge-Kutta 2 (RK2), alternative Runge-Kutta 2 (RK2*), and Runge-Kutta 4 (RK4)[18] methods, are employed to solve the differential equations. Based on these solutions, predictions about the COVID-19 pandemic for the upcoming year are made.

In the BVP section, a linear spring model (LSM) is established. This model extends a small set of measurement data using simulation algorithms and calculates the spring stiffness coefficient through linear regression, a machine learning algorithm[28]. The displacement of the elastic body under force distribution, representing force density, is measured[23]. Numerical solutions are obtained using forward, central, and backward finite difference methods (FDM)[7][8], the Galerkin method in variational methods (VGLM)[9], the collocation method (CD)[27], and the Galerkin method in finite element methods (FEM)[10][11]. The models are evaluated using mean squared error (MSE)[21].

Finally, the analytic solutions for both models are provided and compared with the numerical solutions to assess their accuracy and efficiency[25]. In this paper, the analytical solution after

Laplace transform is given[17].This comparison highlights the strengths and weaknesses of each numerical method in solving real-world problems, providing valuable insights for future research and applications.

1.1. Initial value problem application

The first part of the paper concentrates on analyzing the relationship between the number of infected, recovered, and susceptible individuals in India during the COVID-19 pandemic by establishing an SIR epidemic model and predicting the future infection trends in the region. This section serves as a practical application of initial value problems in ODEs. During this process, we employed machine learning techniques, dividing the dataset into training and testing sets, pre-training the model on the training set, and then evaluating the parameter performance on the testing set. After training, the trained parameters were applied to the model.

To develop the SIR model, we collected relevant data on COVID-19 in India, including the number of confirmed cases, recoveries, and deaths from authoritative sources such as the Ministry of Health and Family Welfare, India, and the World Health Organization (WHO). We preprocessed the collected data to ensure its consistency and accuracy. The key model parameters, including the transmission rate (β) and the recovery rate (γ), were estimated using machine learning methods[15][16]. This paper divided the dataset into training and testing sets, pre-trained the model parameters using the training set, and validated the parameter performance on the testing set to ensure the robustness and generalization ability of the model.

After parameter estimation, this paper employed various numerical methods to solve the SIR model. These methods included first-order and second-order Taylor series expansions, the Euler method, the improved Euler method, RK2, RK2 optional, and RK4 methods. These methods approximated the solutions of the model using different numerical integration techniques. Each method's performance in solving the SIR model was compared to assess their accuracy and efficiency. The results showed that each method had its strengths and weaknesses in predicting the number of susceptible, infected, and recovered individuals.

By comparing the results of different numerical methods, it could evaluate which method was more effective and accurate in solving the SIR model. The results theoretically indicated that the Runge-Kutta methods, especially the RK4 method, excelled in balancing computational complexity and solution accuracy. The findings of this study provide important insights into understanding the dynamics of infectious diseases and offer valuable information for public health planning and intervention strategies.

In summary, this study combines machine learning techniques and numerical methods to analyze the spread of COVID-19 in India, demonstrating the practical application of initial value problems in ODEs. The study results highlight the importance of integrating different approaches in epidemiological modeling, further enriching understanding of infectious disease dynamics.

1.2. Boundary value problem application

This article aims to introduce and compare six methods for solving the linear spring model, including three finite difference methods (forward difference method, central difference method, and backward difference method), the Galerkin method in variational methods, the collocation method in variational methods, and the Galerkin method in finite element methods. By using these different solution methods, this paper can evaluate the suitability of each method for this model.

In this section, the article focuses on addressing boundary value problems. A suitable theme here is to explore the variation of force density at a specific position of a linear spring. The article mainly considers a spring fixed at both ends, which is in a state of static equilibrium when subjected to an external force. The physical meaning of the linear spring model indicates that the second derivative of the spring at position x, i.e., the degree of bending, is equal to the ratio of the applied external force density to the spring's stiffness coefficient k.

In this context, the article emphasizes six methods, including three finite difference methods (forward, central, and backward), the Galerkin method in variational methods, the collocation method in variational methods, and the Galerkin method in finite element methods. By using different solution methods, it can evaluate the suitability of each method in this model.

The forward difference method focuses on solving time-dependent problems and is usually used for time-dependent problems. It is easy to implement and suitable for beginners. However, when dealing with boundary value problems, the numerical stability of the forward difference method may not be sufficient, and boundary conditions need to be handled carefully. The central difference method provides higher accuracy and symmetry, making it suitable for symmetric problems. This method uses a symmetric difference scheme, which offers better numerical stability, but caution is still needed when handling boundary conditions in some cases. The backward difference method provides higher numerical stability when dealing with certain types of boundary value problems and is often used in implicit numerical methods. It is suitable for solving stiff problems and specific boundary value problems.

The Galerkin method in variational methods is based on the variational principle and obtains the solution of the differential equation by solving the extremum of the functional. It is suitable for problems with complex geometries and boundary conditions, providing high-precision numerical solutions. The collocation method, by selecting appropriate collocation points and basic functions, can provide high-precision numerical solutions, suitable for nonlinear and complex boundary conditions, with good numerical stability. The Galerkin method in finite element methods divides the solution region into multiple small regions and approximates the solution in each element using simple functions. It can handle complex geometries and boundary conditions, providing high-precision and stable numerical solutions, and has strong adaptability, making it suitable for solving complex practical engineering problems.

By comparing these six methods, we can find that each method has different focuses and applicable ranges when solving the linear spring model. The finite difference methods are simple and easy to implement, suitable for beginners and problems with regular grids. The Galerkin and collocation methods in variational methods are suitable for complex geometries and boundary conditions, providing high-precision numerical solutions. The Galerkin method in finite element methods has strong adaptability, capable of handling complex practical engineering problems. Through practical solutions and comparisons, we can better understand the advantages and disadvantages of each method and choose the numerical solution method suitable for specific problems.

2. Methodology

2.1. Initial value problem models

In the second part of this paper, we provide a detailed introduction to several numerical methods used to solve initial and boundary value problems for ordinary differential equations (ODEs). These methods include Taylor method, the Euler method, the improved Euler method, the second order Runge-Kutta method (RK2), the optional second order Runge-Kutta method (RK2 optional), and the fourth order Runge-Kutta method (RK4). Additionally, we explore three finite difference methods and the Galerkin method and collocation method in the context of variational methods.

Taylor method(TM) is a powerful mathematical tool that approximates the value of a function by expanding its value at a point into a series of polynomials. Differential equations can be estimated and solved effectively by using Taylor expansion.

$$f(x) = f(a) + f(a)'(x-a) + \frac{f''(a)}{2!}(x-1)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

The Euler method(EM) is a fundamental numerical integration technique that approximates the solution of an ODE by discretizing the time interval and using the derivative at the current time step to estimate the value at the next time step. Specifically, given the initial value $y(t0) = y0$ and step size, the Euler method formula is:

$$y_{n+1} = y_n + hf(t_n + y_n)$$

where tn and yn are the time and approximate solution at step n. While the Euler method is simple to implement, its linear approximation nature results in relatively low accuracy and significant error.

To improve accuracy, this essay employs the modefied Euler method(ME), also known as the Heun method. This method calculates an initial prediction and then corrects it to enhance accuracy. The formula is:

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

This method reduces error by introducing a correction step, offering better accuracy than the basic Euler method.

The second order Runge-Kutta method (RK2) further improves accuracy by performing two intermediate calculations at each time step. Its formula is:

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

This method strikes a good balance between computational complexity and accuracy.

The optional second order Runge-Kutta method (RK2 optional) provides an alternative approach with different intermediate calculations. The formula is:

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$y_{n+1} = y_n + \frac{1}{4}(k_1 + 3k_2)$$

This variant offers increased flexibility through different weighting combinations.

The fourth order Runge-Kutta method (RK4) significantly enhances accuracy by performing four intermediate calculations at each time step. The formula is:

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

The RK4 method is one of the most used numerical methods due to its optimal balance between computational complexity and solution accuracy.

The SIR model is a fundamental epidemiological model used to understand and predict the spread of infectious diseases within a population. Developed by Kermack and McKendrick in 1927, the model divides the population into three compartments: Susceptible, Infected, Recovered. Susceptible is individuals who are vulnerable to contracting the disease. Infected are individuals who have contracted the disease and can spread it to susceptible individuals. Recovered are individuals who have recovered from the disease and are now immune, or who have been removed from the susceptible population due to death or other reasons. The model is based on a set of ordinary differential equations (ODEs) that describe the rate of change in the number of individuals in each compartment over time. The interactions between these compartments are governed by two key parameters: β and γ. β is the rate at which susceptible individuals become infected upon contact with an infected individual. γ is the rate at which infected individuals recover and move to the recovered compartment.

The dynamics of the SIR model are described by the following equations:

$$\begin{cases} \frac{dS}{dt} = -\beta\frac{SI}{N} \\ \frac{dI}{dt} = \beta\frac{SI}{N} - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases}$$

2.2. Boundary value problem models

Firstly, the article establishes a linear spring model based on Hooke's Law, which describes the behavior of the spring, such as stretching or compressing within its elastic limit. This model is formulated as a second-order ordinary differential equation:

$$\frac{d^2u(x)}{dx^2} = -\frac{f(x)}{k}$$

Here f(x) is the force density of the spring, u(x) is position function of spring and k is the stiffness factor.

The three finite difference methods(FDM) include forward difference, backward difference, and central difference. These methods approximate derivative values using different finite difference formulas. The forward difference uses the current and next function values:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

The backward difference uses the current and previous function values:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

The central difference uses the average of the function values at the points before and after the current point:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

These difference methods are particularly useful in solving boundary value problems by providing different approximation accuracies and stabilities through various discretization techniques.

In the context of variational methods, the Galerkin method(VGLM) is a common numerical technique that approximates the solution by representing it as a linear combination of basic functions and ensuring that the residual is zero in a weighted inner product sense. The steps involve selecting appropriate basis functions, constructing a trial solution, and solving for the approximate solution using the weighted residual approach.

$$u_r(x) = \begin{cases} x^{r-1}(x-a)(x-b) \\ \sin\left[\frac{r\pi(x-a)}{b-a}\right] \end{cases} \text{ depending on the given problem}$$

$$R(x,c_r)=[p(x)\frac{d^2}{dx^2}+q(x)\frac{d}{dx}+r(x)]\sum_{r=1}^{n}c_r u_r(x)-g(x)$$

$$\int_a^b u_r(x)R(x,c_r)dx=0, for\ r=1,2,\ldots,n.$$

The collocation method(CD) is a discretization technique that approximates the differential equation directly at a set of discrete points, forming a system of algebraic equations. Solving these equations yields the approximate solution. This method ensures the differential equation is satisfied at specific collocation points, providing a direct and efficient numerical solution approach. The collocation method is highly effective in solving complex boundary value problems, offering high-precision numerical solutions.

$\int_a^b Re\ \delta(x-x_i)\,dx=0, i=1,2,\ldots,n.$ Re is the residual function

$$\frac{d^2y}{dx^2}+y=\delta(x-a)$$

The Finite Element Method (FEM) is a powerful numerical technique used for solving boundary value problems (BVPs) in various fields such as engineering, physics, and applied mathematics. One of the prominent approaches within FEM is the Galerkin method, which ensures the solution satisfies the governing equations in a weighted integral sense. The Galerkin method is known for its ability to provide accurate and stable solutions to complex problems.

$$\frac{dy}{dx}\left[a(x)\frac{dy}{dx}\right]=-f(x),\quad a\le x\le \beta$$

$$y(x)=\sum\nolimits_e y^{(e)}(x)$$

$$y^{(e)}=\sum_{i=1}^{2}L_i{}^{(e)}(x)y_i{}^{(e)}$$

$$L_1{}^{(e)}(x)=\frac{x_e-x}{x_e-x_{e-1}}\qquad L_2{}^{(e)}(x)=\frac{x-x_{e-1}}{x_e-x_{e-1}}$$

$$L_i{}^{(e)}(x_j)=\begin{cases}0 & i\ne j\\ 1 & i=j\end{cases},\quad \sum\nolimits_{i=1}^{2}L_i{}^{(e)}(x)=1$$

$$\int_{x_{e-1}}^{x_e}L_i(x)\left[\frac{d}{dx}\left(a\frac{dy}{dx}\right)+f(x)\right]dx=0,\quad i=1,2,$$

$$\begin{bmatrix}K_{11}^{(e)} & K_{12}^{(e)}\\ K_{21}^{(e)} & K_{22}^{(e)}\end{bmatrix}\begin{bmatrix}y_1^{(e)} & .\\ y_2^{(e)} & .\end{bmatrix}=\begin{bmatrix}F_1^{(e)} & .\\ F_2^{(e)} & .\end{bmatrix}$$

$$KY=F$$

$$K=\begin{bmatrix}K_{11}^{(1)} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & K_{22}^{(n)}\end{bmatrix}$$

$$Y=\begin{pmatrix}y_0 & \cdots & .\\ \vdots & \ddots & \vdots\\ y_n & \cdots & .\end{pmatrix},\quad F=\begin{pmatrix}F_1^{(1)} & \cdots & .\\ \vdots & \ddots & \vdots\\ F_2^{(n)} & \cdots & .\end{pmatrix}$$

In summary, these numerical and variational methods provide a range of tools for solving initial and boundary value problems for ODEs. By choosing and combining different methods, we can balance computational complexity and solution accuracy, effectively addressing practical problems.

2.3. Analytic Solution

In the realm of mathematics and engineering, finding solutions to equations is a fundamental task. Among the various methods of solving equations, analytical solutions hold a prominent place due to their precision and explicitness. An analytical solution, also known as an exact solution, is a solution to a mathematical equation that can be expressed in terms of elementary functions such as polynomials, exponential functions, logarithms, and trigonometric functions. This paper aims to delve into the nature, characteristics, applications, and limitations of analytical solutions, providing a comprehensive overview of this essential concept.

An analytical solution is characterized by its ability to provide a clear and explicit formula for the solution of an equation. Unlike numerical solutions, which approximate the solution through iterative methods, analytical solutions offer exact expressions that can be directly used to calculate the values of the solution. This exactness is particularly valuable in theoretical analysis, where precision is paramount.

Analytical solutions are distinguished by several key characteristics. The explicit nature of analytical solutions means that the results are presented in a clear and understandable formula, facilitating direct computation and analysis. The accuracy of analytical solutions is another notable feature, as they are exact, eliminating the approximation errors inherent in numerical methods. This accuracy is crucial in applications where even small errors can lead to significant consequences. Furthermore, once derived, an analytical solution can be applied to any situation that satisfies the same initial or boundary conditions, making analytical solutions versatile and widely applicable. Additionally, analytical solutions can be easily verified by substituting them back into the original equation, ensuring the correctness of the solution and building confidence in its validity.

The utility of analytical solutions spans various scientific and engineering disciplines. In physics, analytical solutions are used to describe the motion of physical systems, solve heat conduction equations, wave equations, and more. For example, the motion of a simple harmonic oscillator can be exactly described by an analytical solution. In engineering, analytical solutions provide precise models essential for design and analysis in structural mechanics, fluid mechanics, and circuit analysis. For instance, the stress distribution in a beam under load can be determined using analytical methods. In mathematics, the study of differential equations, integral equations, and algebraic equations heavily relies on analytical solutions, offering insights into the behavior of mathematical models and facilitating further theoretical developments.

Several methods are employed to derive analytical solutions. Algebraic methods involve techniques such as factorization, completing the square, and using the quadratic formula to solve algebraic equations. Calculus methods include separation of variables, integrating factors, and transformation techniques for solving differential equations. For example, the method of separation of variables is commonly used to solve partial differential equations. In some cases, analytical solutions are expressed using special functions like Bessel functions, Legendre polynomials, and others, extending the range of problems that can be solved analytically.

2.3.1 Laplace Transformation

The Laplace transform is a powerful mathematical tool used to convert complex differential equations into simpler algebraic equations, facilitating the process of finding analytical solutions. This transformation technique is widely used in various fields such as engineering, physics, and applied mathematics due to its ability to handle linear time-invariant systems and provide precise solutions. This part aims to explore the nature, applications, and benefits of the Laplace transform as an analytical solution method.

The Laplace transform is an integral transform that converts a time-domain function $f(t)$ into a complex frequency-domain function $F(s)$. The transform is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt$$

where: t is the time variable.

s is the frequency variable, $s = \sigma + iw$

f(t) is the original function in the time domain.

F(s) is the transformed function in the frequency domain.

Next, the analytical solutions of the two models in this paper are obtained by using Laplace changes.

2.3.2 Laplace Transform in SIR model

Suppose the initial value at t = 0 s, the number of the Susceptible is $S(0) = S_0$, the number of the Infectious is $I(0) = I_0$, the number of the Recovered is $R(0) = R_0$.

For first equations in SIR model it has:

$$\mathcal{L}\left(\frac{dS}{dt}\right) = \mathcal{L}(-\beta SI)$$

According to Properity of the Laplace Transform $\mathcal{L}\left(\frac{dS}{dt}\right) = s\,S(s) - S_0$:

$$sS(s) - S_0 = -\beta\mathcal{L}(SI)$$

Same procedure for second and third equations in SIR model:

$$\mathcal{L}\left(\frac{dI}{dt}\right) = \mathcal{L}(\beta SI - \gamma I)$$

$$\mathcal{L}\left(\frac{dI}{dt}\right) = s\,I(s) - I_0$$

$$sI(s) - I_0 = \beta\mathcal{L}(SI) - \gamma I(s)$$

Third equation :

$$\mathcal{L}\left(\frac{dR}{dt}\right) = \mathcal{L}(\gamma I)$$

$$\mathcal{L}\left(\frac{dR}{dt}\right) = s\,R(s) - R_0$$

$$sR(s) - R_0 = \mathcal{L}(\gamma I)$$

It shows new system of equations:

$$\begin{cases} sS(s) - S_0 = -\beta\mathcal{L}(SI) \\ sI(s) - I_0 = \beta\mathcal{L}(SI) - \gamma I(s) \\ sR(s) - R_0 = \mathcal{L}(\gamma I) \end{cases}$$

Solve them using Laplace Transform. There are some requirments or hypothesis that S and I won't change quickly at intial value, then it has condition that $S(t) \approx S_0$ and $\mathcal{L}(SI) = S_0 I(s)$

$$\begin{cases} S(s) = \dfrac{S_0}{s + \beta S_0 I(s)} \\ I(s) = \dfrac{I_0}{s + \gamma - \beta S_0} \\ R(s) = R_0 + \gamma \dfrac{R_0}{s(s + \gamma - \beta S_0)} \end{cases}$$

Using the Inverse Laplace Transform:

$$I(t) = \mathcal{L}^{-1}\left\{\frac{I_0}{s + \gamma - \beta S_0}\right\} = I_0 e^{-(\gamma - \beta S_0)t}$$

$$R(t) = \mathcal{L}^{-1}\left\{R_0 + \gamma \frac{I_0}{s} * \frac{1}{s + \gamma - \beta S_0}\right\}$$

Using partial fraction factorization and inverse Laplace transform table:

$$R(t) = R_0 + \gamma I_0 \mathcal{L}^{-1}\left\{\frac{1}{s(s + \gamma - \beta S_0)}\right\}$$

$$R(t) = R_0 + \gamma I_0 (\frac{1}{\gamma - \beta S_0}(1 - e^{-(\gamma - \beta S_0)t}))$$

Same procedure as R(t):

$$S(t) \approx S_0 - \beta S_0 \int_0^t I(\tau) d\tau$$

Where τ is the time variable.

2.3.3 Laplace Transform in LSM model

Take $\mathcal{L}\left\{\frac{d^2u(x)}{dx^2}\right\} = \mathcal{L}\left\{-\frac{f(x)}{k}\right\}$
where $\mathcal{L}\{u(x)\} = U(s)$

$$\mathcal{L}\left\{\frac{d^2u(x)}{dx^2}\right\} = s^2U(s) - su(0) - \frac{du(0)}{dx}$$

$$\mathcal{L}\{f(x)\} = F(s)$$

$$s^2U(s) - su(0) - \frac{du(0)}{dx} = -\frac{1}{k}F(s)$$

Then find the *U(s):*

$$U(s) = \frac{-\frac{1}{k}F(s) + su(0) + \frac{du(0)}{dx}}{s^2}$$

Then take the partial factorization:

$$U(s) = -\frac{1}{k}\frac{F(s)}{s^2} + \frac{u(0)}{s} + \frac{\frac{du(0)}{dx}}{s^2}$$

Take Inverse Laplace Transform:

$$u(x) = \mathcal{L}^{-1}\{U(s)\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = x$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 1$$

Finally get:

$$u(x) = -\frac{1}{k}\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} + u(0)x + \frac{du(0)}{dx}x$$

$$u(x) = -\frac{1}{k}\int_0^x F(\tau)d\tau + u(0)x + \frac{du(0)}{dx}x$$

This solution represents the displacement *u(x)* of the elastomer under the force density distribution *f(x)*.

3. Experiments and Results

3.1. IVP Problem

3.1.1 Data preprocessing

This paper uses a dataset of population changes in India during the COVID-19 pandemic. The main data sources are daily infections, daily recoveries, total infections, and total recoveries. Firstly, the data is preprocessed, and the missing value and duplicate value are checked first.

In this paper, the linear interpolation method is used to deal with the problem of missing values in data sets[22].

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{(x_2 - x_1)}$$

This article deals with the problem of duplicate values by combining duplicate information[24].

$$Combined\ Value = Aggregation\ Function(x_1, x_2, \cdots, x_n)$$

3.1.2 Data Visualization

In the model, the three most important parameters are the number of susceptible people, the number of infected people and the number of recovered people. The paper prints the data of these three parameters over time and estimates the trend of the data roughly by observing the images.

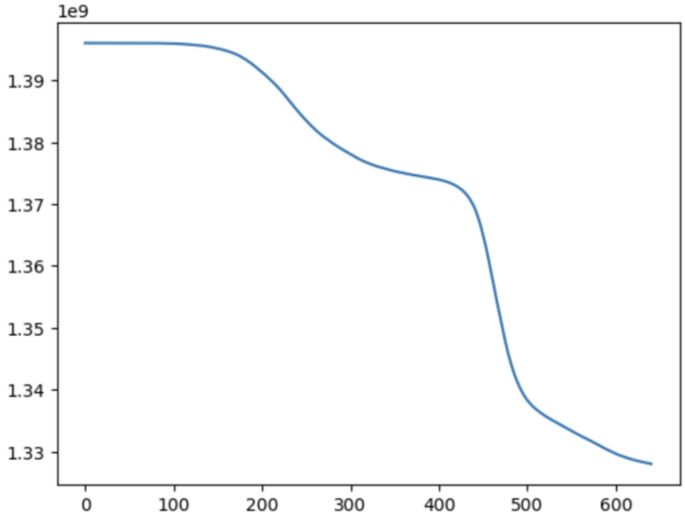

Figure 1. Data trend

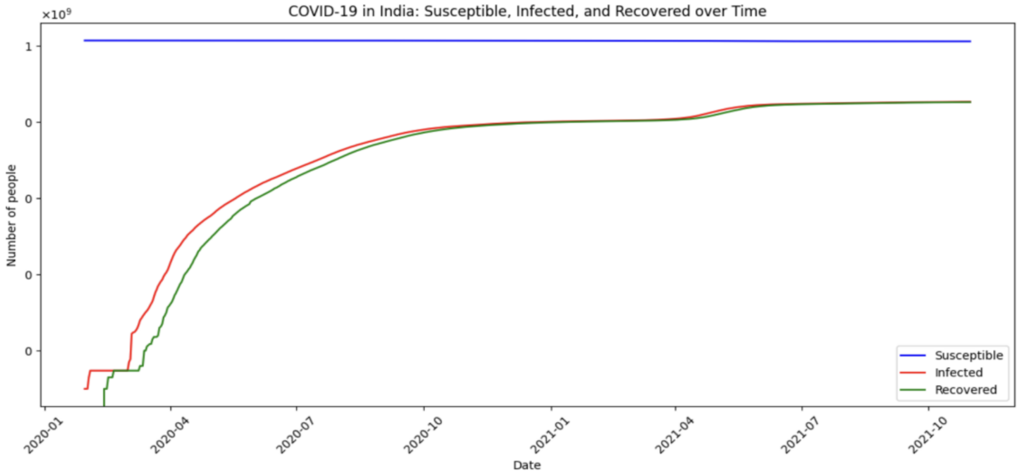

Figure 2. COVID-19 in India: Susceptible, Infected, and Recovered over Time

Because the order of magnitude of the number of susceptible persons is particularly large compared to the number of infected persons and the number of recovered persons, this paper prints the trend of the number of susceptible persons separately. Of course, it can clearly see that the number of people recovering, and the number of people infected have gradually increased over time, and the two have fluctuated with each other.

3.1.3 Training parameters using Machine Learning

Before training the model, the SIR Model is established. The previous section has calculated the number of Susceptible Infected and Recovered. Therefore, in the experimental part, this paper reads the specific columns of the data set as the data of the above three items. Then, the data set is segmented. The training set and the test set are assigned in a 7:3 ratio.

$$D_{train} : D_{test} = 7:3$$

Then the most important part, this paper uses genetic algorithm to optimize the parameters. Genetic algorithm is a kind of optimization algorithm that simulates the process of natural selection, and finds the optimal solution through selection, crossover, and mutation.

Parents are selected according to the individual's fitness, and the selection probability is proportional to the fitness. Common selection methods include roulette selection, sorting selection and tournament selection. Here is the selection part:

$$P(i) = \frac{f(x_i)}{\sum_{j=1}^{N} f(x_i)}$$

where f(x) is the fitness function.

The selected parents are crossed to generate new individuals (progeny). The crossover method includes single-point crossover, multi-point crossover and uniform crossover. Here we use a single point crossing:

$$Child_1 = Parent_1[:c] + Parent_2[c:]$$

$$Child_2 = Parent_2[:c] + Parent_1[c:]$$

Here this part uses the mean square error MSE as the error function. Then the optimization parameter method is used to make the MSE as small as possible, and the optimal β and γ values are obtained under this condition.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y})^2$$

Here's how to use genetic algorithms to optimize parameters.

A random initial population containing beta and gamma is first generated, and then the error function value for everyone in the population is calculated by defining a fitness function equal to the negative mean square error. Individuals are then selected for the next generation based on the previously calculated fitness value, usually using roulette selection or tournament selection. Then, the new individual is generated by crossing operation, and the parent characteristics are mixed. The uniform crossing method is adopted here to achieve the crossing operation. Finally, the variation operation simulates population diversity by introducing random changes. Repeat the above steps until the maximum number of iterations is reached or the error function converges.

The good news is that to better fit the transmission situation in different stages of the epidemic development process, this paper adopts the dynamic adjustment parameter method to divide the time period into three segments and optimize the parameters of the SIR Model in each segment.

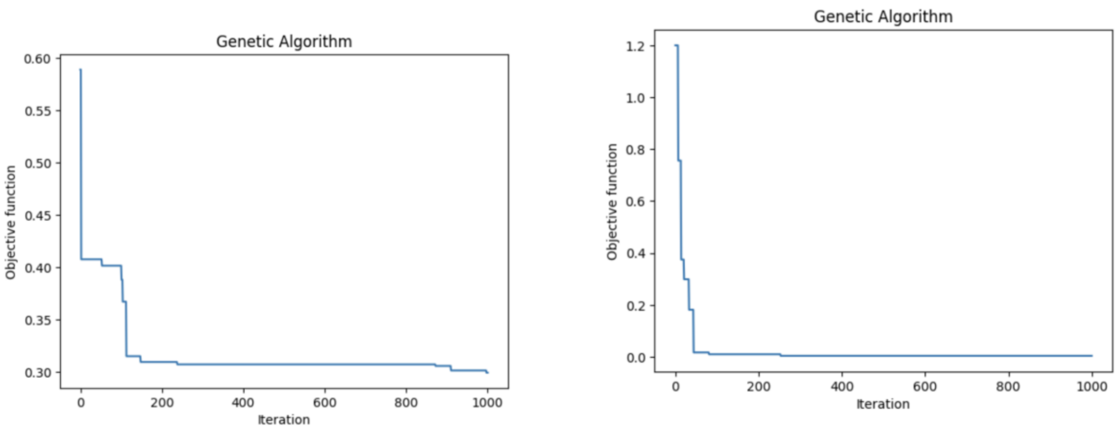

Figure 3. Genetic Algorithm

The values of β and γ are obtained by iterative image of genetic algorithm. Beta and gamma are equal to about 10 percent, and the specific value will oscillate up and down. The image shows that in the training set, the number of infected people fluctuates with the actual value. In the normal model, with the increase of time, the number of susceptible people will slowly move to the number of infected people, and the number of infected people will slowly move to the number of recovered people, forming a dynamic balance. The image proves it. In the test set, the difference between the predicted value and the actual value of the model is almost negligible, which proves the effectiveness of the parameter training. Of course, the values of the mean square error of the training set and the verification set are provided.

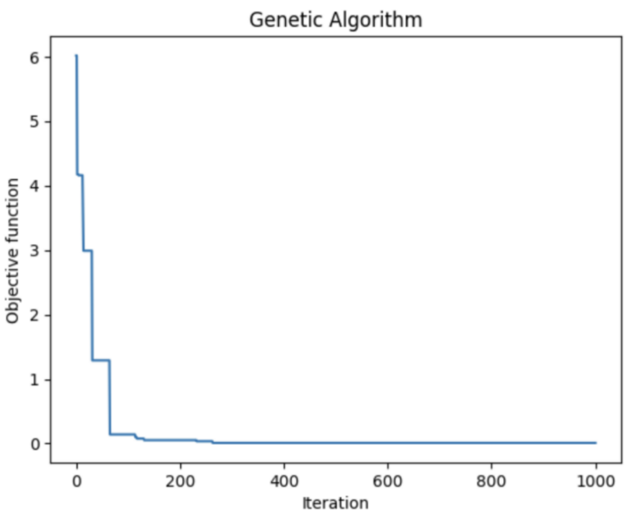

Figure 4. Genetic Algorithm

```
Train MSE: 0.10430367945964797
Test MSE: 0.2296918046608267
```

The image shows that the mean square error of the test set and the verification set is not much different, which proves that the parameters trained by the machine learning model are in line with the actual problem.

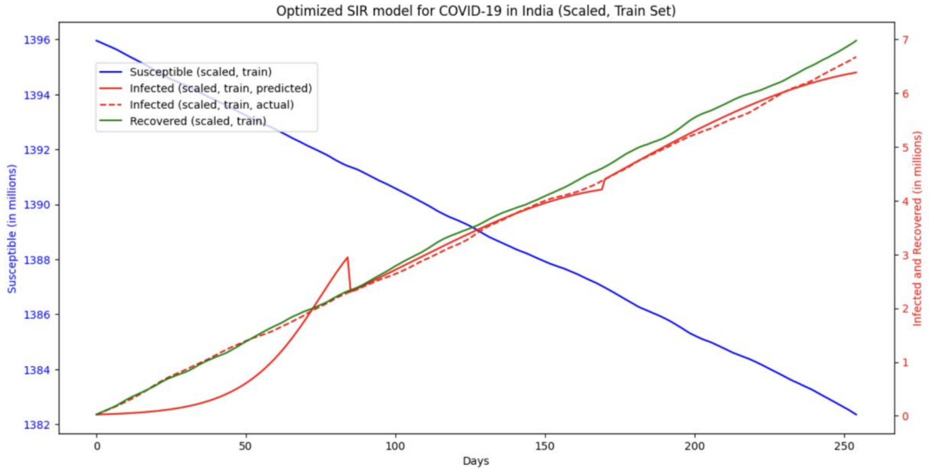

Figure 5. Optimized SlR model for COVlD-19 in India (Scaled, Train Set)

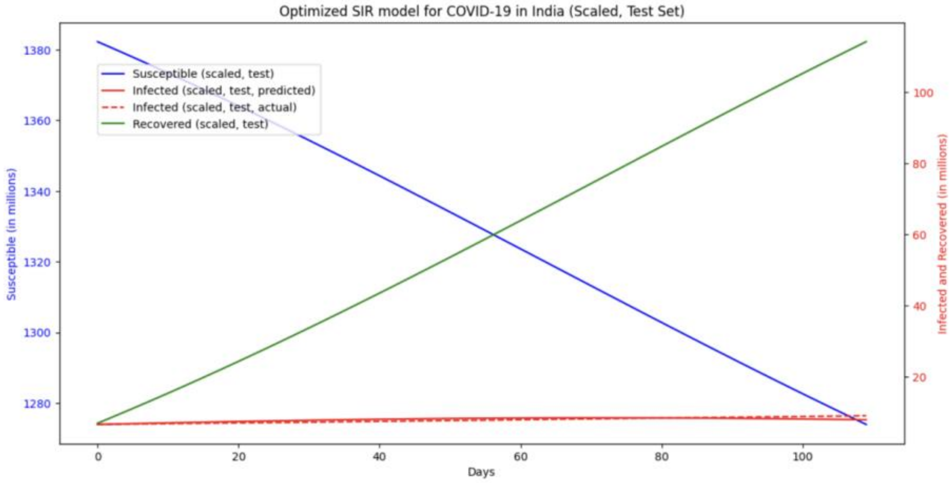

Figure 6. Optimized SIR model for COVlD-19 in India (Scaled, Test Set)

Here, this paper gives a comparison table of six numerical solutions and analytical solutions. It can be seen that in a certain order of magnitude, the difference between the six methods is not particularly significant, but according to the actual data, RK4 is still the best performance.

3.1.4 Prediction based on the trained parameters

SIR Model parameters β and γ were obtained by training in the third stage. This part focuses on the predictive performance of the model. The data set is up to October 31, 2021, so the predictions in this paper are based on October 31, 2021. The paper made year-long predictions. This prediction problem is essentially an initial value problem of differential equations. Now, the unknowns of the model including the initial value conditions and the model parameters have been confirmed. Then the prediction is made in this paper, and the numerical method of differential equation is used to solve it. The numerical methods here mainly include six kinds. These are Taylor's first and second order expansions. Euler's method, improved Euler's method, RK2, RK2 Optional and RK4 methods. These numerical methods play an important role in solving the model. At the same time, this part compares the solving ability of each method on the model through visualization.

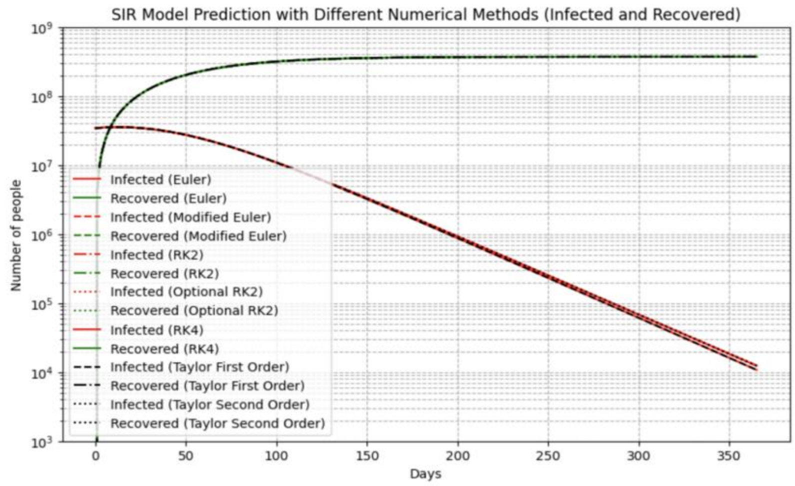

Figure 7. SIR Model Prediction with Different Numerical Methods (infected and Recovered)

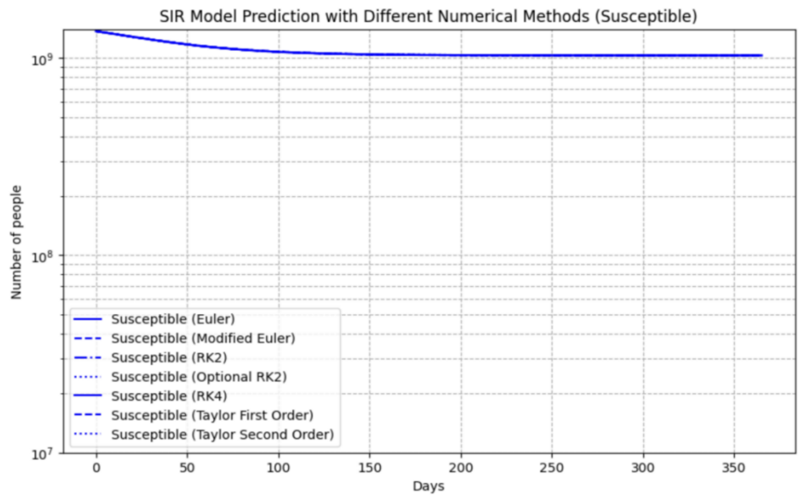

Figure 8. SIR Model Prediction with Different Numerical Methods (Susceptible)

Through the image, in the general direction, the six methods are basically the same. The overall picture describes a trend of increasing numbers of people recovering and decreasing numbers of people infected, which may be related to the development of COVID-19 vaccines and related policies, or some theories about herd immunity can also explain this.

Of course, in the code part, the paper also prints out the analytical solution and numerical solutions of the six methods.

Table 1. The numerical and analytical solutions in COVD-19

Days	TM1	TM2	EM	ME	RK2	RK2*	RK4	LT
0	1.37	1.37	1.37	1.37	1.37	1.37	1.37	1.36
30	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.28
60	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.20
90	1.08	1.09	1.08	1.09	1.09	1.09	1.09	1.14
120	1.05	1.06	1.05	1.06	1.06	1.06	1.06	1.10
150	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.07
180	1.03	1.04	1.03	1.04	1.04	1.04	1.04	1.05
210	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.04
240	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
270	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
300	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
330	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
360	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03

A. Actual value is excel value mutiplied by 1 billion
B. This table meausred the number of the Susceptible

Table 2. The numerical and analytical solutions in COVD-19

Days	TM1	TM2	EM	ME	RK2	RK2*	RK4	LT
0	343.00	343.00	343.00	343.00	343.00	343.00	343.00	343.00
30	335.00	333.00	335.00	333.00	333.00	333.00	333.00	333.10
60	237.00	236.00	237.00	236.00	236.00	236.00	236.00	235.00
90	135.00	135.00	135.00	135.00	135.00	135.00	135.00	135.00
120	67.80	68.90	67.80	68.90	68.90	68.90	68.90	68.92
150	32.10	33.10	32.10	33.10	33.10	33.10	33.10	33.10
180	14.80	15.50	14.80	15.50	15.50	15.50	15.50	15.53
210	6.69	7.12	6.69	7.12	7.12	7.12	7.12	7.12
240	3.02	3.26	3.02	3.26	3.26	3.26	3.26	3.26
270	1.36	1.49	1.36	1.49	1.49	1.49	1.49	1.49
300	0.61	0.68	0.61	0.68	0.68	0.68	0.68	0.68
330	0.27	0.31	0.27	0.31	0.31	0.31	0.31	0.31
360	0.12	0.14	0.12	0.14	0.14	0.14	0.14	0.14

A. Actual value is excel value mutiplied by 100 thousand
B. This table meausred the number of the Infectious

Table 3. The numerical and analytical solutions in COVD-19

Days	TM1	TM2	EM	ME	RK2	RK2*	RK4	LT
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30	1.28	1.28	1.28	1.28	1.28	1.28	1.28	1.28
60	2.35	2.33	2.35	2.33	2.33	2.33	2.33	2.33
90	3.03	3.00	3.03	3.00	3.00	3.00	3.00	3.00
120	3.39	3.37	3.39	3.37	3.37	3.37	3.37	3.37
150	3.57	3.54	3.57	3.54	3.54	3.54	3.54	3.54
180	3.65	3.63	3.65	3.63	3.63	3.63	3.63	3.63
210	3.69	3.67	3.69	3.67	3.67	3.67	3.67	3.67
240	3.70	3.69	3.70	3.69	3.69	3.69	3.69	3.67
270	3.71	3.70	3.71	3.70	3.70	3.70	3.70	3.67
300	3.72	3.70	3.72	3.70	3.70	3.70	3.70	3.69
330	3.72	3.70	3.72	3.70	3.70	3.70	3.70	3.70
360	3.72	3.70	3.72	3.70	3.70	3.70	3.70	3.71

A. Actual value is excel value mutiplied by 100 milion
B. This table meausred the number of the Recovered

3.2. BVP Problem

3.2.1 Simulating Data

Due to the scarcity of experimental data, this paper expands the simulated data reasonably, which makes the experimental data greatly improved. This is done by randomly generating 2000 mass values between 0.1 and 1 through linspace in the numpy package. Then, using the formula tension equals gravity, calculate the force corresponding to each mass, where g is equal to 9.8m/s^2. Finally, the elongation data is generated and random noise with a standard deviation of 0.005 is added to simulate the experimental error.

Data sets containing mass, force, and elongation are read, and then the data set is divided into training sets and test sets.

$$F = kx$$

According to Hooke's law k = F/x, the value of k obtained by training is about 50.32, and the mean square error is only about 0.07, which is a good experimental result.

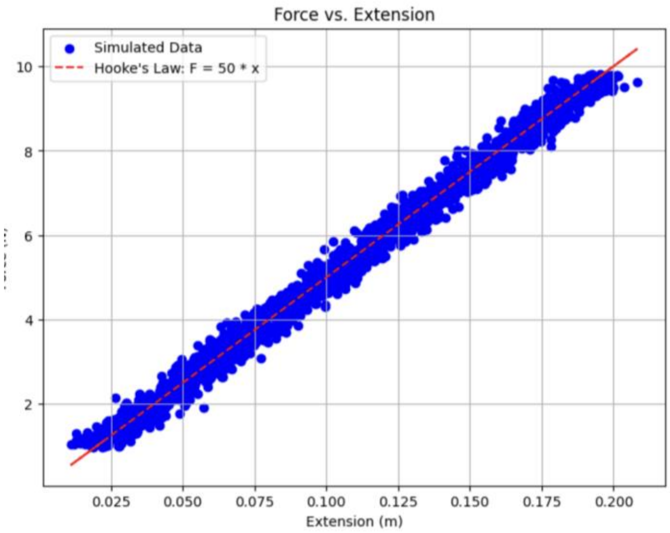

Figure 9. Force vs. Extension

3.2.2 Interpretation of Force Density

In this part, the force density of different positions is obtained by interpolating the displacement and force of different positions in the data. The force density f(x) is the force per unit length. The experiment requires the force density on the discrete nodes, so the interpolation operation is realized by linear interpolation method in this paper. Linear interpolation is a simple and commonly used interpolation method, its basic idea is to estimate the value of the unknown data points through the linear relationship between the known data points. For two known points (x1, y1) and (x2, y2), the interpolation point (x,y) is calculated by the formula

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{(x_2 - x_1)}$$

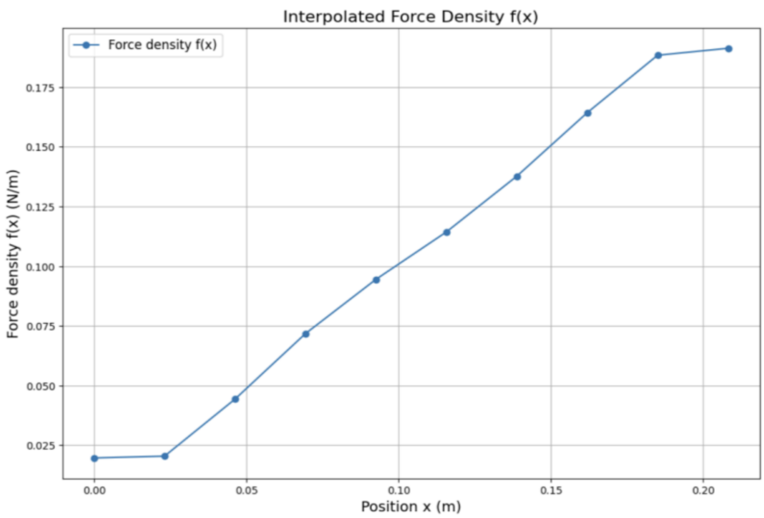

Figure 10. Interpolated Force Density f(x)

Finally, the interpolated force density image is obtained. From the image, the force density changes slowly in the initial position and the end position, and changes faster in the middle part, and the force density of the independent variable is positively correlated with that of the dependent variable.

3.2.3 Solve the Spring Model using six methods

In this part, the linear model of the spring(LSM) is solved. In Part 2.2, the paper introduces the spring linear model, which mainly describes the stretching and compression behavior of the spring within the elastic limit. Therefore, this paper uses the spring linear model to further explore the relationship between force density and position. This is essentially a second order differential equation. And the boundary conditions are usually set to 0 at both ends, which means that the spring is fixed at both ends.

Here six methods are used, including forward finite difference, central finite difference, backward finite difference, Galerkin method in variational method, collocation method in variational method and Galerkin method in finite element method.

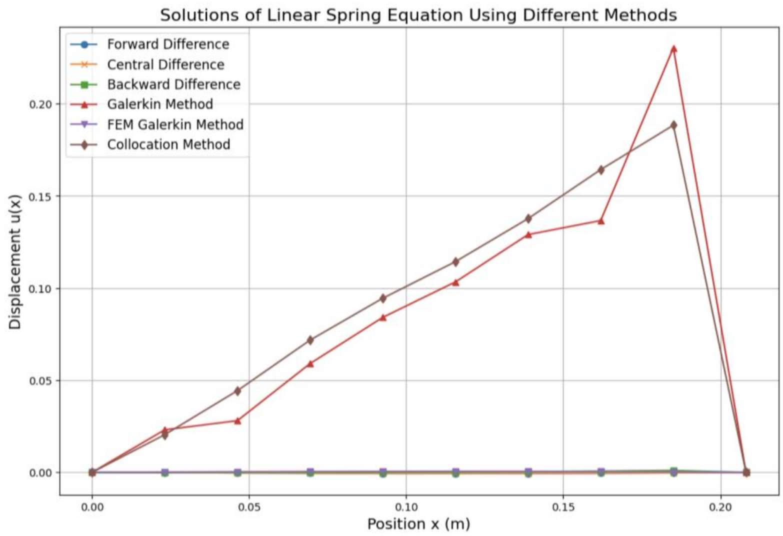

Figure 11. Solutions of Linear Spring Equation Using Different Methods

According to the experimental images, the backward finite difference, the Galerkin method in the variational method, the collocation method in the variational method and the Galerkin method in the finite element show that the force density increases first with the displacement, reaches a critical point and then decreases to 0. The forward difference and central difference mean that they will first decrease to a certain lowest point and then slowly rise to the highest point.

According to the analysis of the experimental results, the spring data is calculated according to the tension, so the increase of force density with the position is reasonable. After reaching a certain point, the force density begins to drop suddenly, which indicates that the force density usually reaches its maximum at a certain position, and the actual situation is that the spring will continue to extend before this position, until the maximum elastic limit is reached, the spring cannot recover the deformation, so as the position continues to extend the force density will continue to decrease and eventually reach 0.

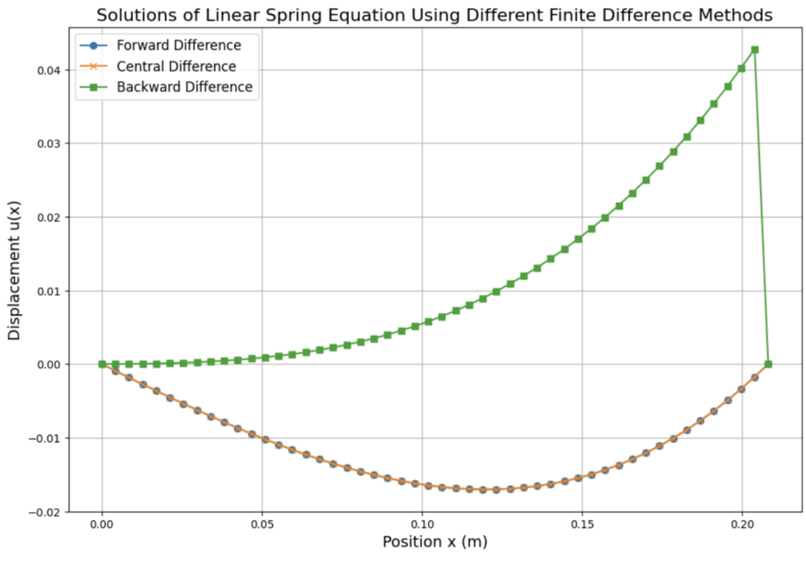

Figure 12. Solutions of Linear Spring Equation Using Different Finite Difference Methods

Therefore, according to the actual situation, the results of forward finite difference and central difference do not conform to the objective law, the two methods show that the linear density of the spring will decrease with the increase of position until it reaches a lowest point, and then increases to 0. This indicates that the actual problem is not suitable for modeling by either method. Of the remaining four methods, according to the image, the Galerkin method of variational method is the most significant. Therefore, this paper chooses it as the baseline, so that it can be used as a control group to measure the gap with other methods. In this paper, the mean square error is used to measure the difference with the Galerkin method of variational method. According to the data, the difference between collocation method and Galerkin method in the absolute error display variational method is the smallest. On the other hand, the remaining methods are similar in terms of absolute error, which is roughly 0.07. Therefore, the elastic deformation failure point of the model is about 0.18.

Finally, this paper gives six methods and analytical solutions in the form of diagrams

Table 4 The numerical and analytical solutions in force density problem (BVP

Position(m)	FDM1	FDM2	FDM3	VGLM	CM	FEM	LT
0	2.37E-18	2.37E-18	-1.50E-18	0.00	0.00	0.00	0.00
0.024	-1.64E-04	-1.64E-04	-1.58E-19	2.31E-02	2.00E-02	1.00E-04	2.32E-02
0.048	-3.18E-04	-3.18E-04	1.09E-05	2.80E-02	4.00E-02	3.10E-04	2.90E-02
0.072	-4.47E-04	-4.47E-04	4.56E-05	5.90E-02	7.00E-02	4.47E-04	5.90E-02
0.096	-5.39E-04	-5.39E-04	1.19E-04	8.40E-02	9.00E-02	5.38E-04	8.40E-02
0.12	-5.79E-04	-5.79E-04	2.42E-04	1.00E-01	1.10E-01	5.79E-04	1.00E-01
0.144	-5.59E-04	-5.59E-04	4.27E-04	1.28E-01	1.37E-01	5.59E-04	1.27E-01
0.168	-4.65E-04	-4.65E-04	6.85E-04	1.36E-01	1.64E-01	4.65E-04	1.34E-01
0.192	-2.83E-04	-2.83E-04	1.03E-03	2.30E-01	1.88E-01	2.83E-04	2.30E-01
0.22	0.00	0.00	0.00	0.00	0.00	0.00	0.00

```
Forward Difference average error: 0.078911975
Central Difference average error: 0.0796753841951
Backward Difference average error: 0.07923181865105
Galerkin Method average error: 0.0
FEM Galerkin Method average error: 0.07900473699999999
Collocation Method average error: 0.013129398000000004
```

4. Conclusion and Limitations

4.1. IVP

In the initial value problem, based on the parameters learned by machine learning, this paper established a SIR Infectious disease model of the novel coronavirus epidemic, predicted by the model, and finally obtained the numerical solution by six numerical methods. The model shows that with the increase of time, the number of rehabilitations will first increase sharply for a short time, and then stabilize. The number of infections will gradually decrease over time. The change trend of the number of susceptible people is not obvious, and it is basically a straight line. So, in the coming year, the infection rate in India will decrease over time, either because of the government's quarantine policy, the spread of the COVID-19 vaccine, or the development of new antibodies from old, infected people that spread in daily life, or because of herd immunity. Either way, the impact of the COVID-19 outbreak is diminishing, and the number of infections will get lower and lower over time. This is exciting news.

Of course, the model still has some limitations. First, the modeling of this model is simple, and only β and γ parameters are considered. It is important to know that a global pandemic is influenced by many factors. SIR Models typically assume that the rates of infection (β) and recovery (γ) are fixed, but in the case of COVID-19, these parameters may change over time and with external conditions. For example, as diseases spread and public health measures are implemented, transmission rates may decrease.

Secondly, in the early stage of model prediction, there will be a drop point, indicating that the generalization ability of the model needs to be improved. Or you need to learn more kinds of infectious disease models to better predict the trend of the model.

Of course, the SIR Model does not consider the incubation period of the disease, which is the time period from when an individual is infected to when they are able to transmit the disease. In some diseases, this incubation period is very important. For example, the SEIR model (Susceptibility - Exposure - infection - Removal model) improves on this by adding an exposure (latency) state.

Finally, the SIR Model assumes that all individuals have an equal opportunity to be exposed to and transmit disease, which means that it assumes that the population is evenly mixed. However, populations are not evenly mixed, and the frequency and pattern of contact between individuals can be influenced by a variety of factors such as geographic location, social behavior, age structure, and more.

4.2. BVP

In the boundary value problem, this paper mainly discusses the effect of the force density of the linear spring with the change of position and conducts the experiment by establishing the linear model of the spring. In the experiment, the data set is trained, and the spring elastic coefficient is obtained. Then, six numerical methods are used to solve second-order ordinary differential equations. The results show that the linear density of the spring increases first with the increase of the elongation and decreases rapidly to 0 after reaching the critical value, which indicates that the spring cannot recover the deformation after reaching the maximum elastic limit. Of course, the numerical methods of backward finite difference, Galerkin method in variational method, collocation method in variational method and Galerkin method in finite element method can effectively estimate the relationship between spring linear density and position.

Of course, there are some limitations to this part of the experiment. The influence of external environment factors such as temperature, humidity and friction on spring performance was not considered in the experiment. These factors may have a significant impact on the experimental results in practical applications.

The numerical method also has some limitations. In the process of numerical solution, the treatment of boundary conditions may affect the accuracy of the results. In this experiment, the processing of the boundary conditions is relatively simple, and more complex methods may be needed to deal with the boundary conditions in practice.

The interpolation method is used in the pre-processing stage, and the interpolation method also has some limitations. In this paper, the linear interpolation method is used to calculate the force density f(x). Although linear interpolation is simple and easy to implement, it may not be accurate enough for cases where data points are sparse or vary dramatically. You can consider using higher-order interpolation methods, such as spline interpolation, to improve accuracy.

Some simplification of the experimental conditions may also have some influence on the experimental results. In the experiment, it is assumed that the spring works under ideal conditions, and the possible non-ideal factors such as the initial preload force and the transverse vibration of the spring are ignored. These factors may affect the results in actual experiments.

Although six numerical methods are used in this paper to demonstrate the solution process and performance comparison of linear spring models, there are still many limitations in the experiment. In practical application, it is necessary to consider the noise model, nonlinear effect, external environmental factors, mesh density, boundary condition treatment, interpolation method selection and experimental conditions to improve the accuracy and robustness of the model. Through further optimization and improvement, the spring linear model can be better understood and applied to provide effective solutions for practical engineering problems.

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