

The Dynamic Evolution of Black Holes in General Relativity: Numerical Relativity Simulation

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Abstract. As the core of the star left behind by the supernova explosion continues to collapse, the immense pressure causes protons to absorb electrons and transform into electrically neutral neutrons. The core of the star eventually forms a dense neutron star about 10 kilometres across, much smaller than a city. Neutron stars resemble giant atomic nuclei and are much denser than any atomic material on Earth. A small cup of matter from a neutron star exceeds the total mass of everyone on Earth. If the mass of the star's core is large enough, it will continue to collapse beyond the Schwarzschild radius, eventually forming a black hole. The gravitational field of a black hole is so strong that neither matter nor electromagnetic waves (including visible light) can escape from it. The boundary of this inescapable region is called the event horizon. In our experiments, we used a high-performance computing cluster and numerical relativity simulation software to accurately solve Einstein's field equations and demonstrate that the peak amplitude of gravitational waves emitted by black holes during mergers can reach 3.1×10^{-21} with a frequency of 300 Hz. This study provides important insights into the dynamic evolution of black holes and the emission of gravitational waves under various initial conditions.

Keywords: Numerical Relativity; Black Hole Evolution; Gravitational Waves; General Relativity; Einstein Field Equations.

1. Introduction

In recent years, with the development of gravitational wave detection technology, especially the successful operation of gravitational wave detectors such as LIGO and Virgo, mankind has for the first time directly detected gravitational waves generated by the merger of two black holes. These observations provide strong evidence to verify general relativity and reveal the dynamic evolution of black holes. However, analytical methods are difficult to accurately describe complex processes such as the merger of double black holes, and numerical relativity has become the main tool to study this field. The aim of this paper is to explore the dynamic evolution of black holes under different initial conditions, including black hole mergers, accretion growth and their gravitational wave emission properties, through numerical relativistic simulations.

2. Related Works

The dynamical evolution of black holes has been specifically studied by physicists for a long time. Ni Y described the evolution of Black holes (BH) and their relation to their host galaxies in Astrid, a high-capacity cosmological hydrodynamical simulation with a box size of 250 h-1Mpc containing 2×55003 particles evolving up to $z = 3$. Astrid statistically models BH gas accretion and the feedback of active galactic nuclei on their environment [1]. Torniamenti S demonstrated direct N-body simulations of two different families of clusters: low-mass clusters ($\sim 500\text{-}800 M_\odot$) and relatively high-mass clusters ($\geq 5000 M_\odot$). It was shown that the pathways of BBH formation in low and high mass clusters were extremely different, leading to two completely different BBH merger populations [2]. Chattopadhyay D evolved a suite of cluster models using the NBODY6 code to study the relationship between the global properties of clusters and the binary black hole populations. He found that the metallicity of the clusters determines their lifetimes and the initial half-mass radius affected the BH exchange interaction rate [3]. Kamlah A W H performed eight direct N-body simulations of

rotating clusters with up to 1.1×10^5 stars using the `nbody6++` GPU. The results showed that rotation and stellar evolution affected the cluster dynamics, leading to initial violent relaxation and gravitational catastrophe, later angular momentum transfer and system collapse [4]. Li R investigated the hydrodynamic evolution of BBHs with different mass ratios, eccentricities and disc properties through high-resolution 2D simulations [5]. Arca Sedda M's interpretation of the observatory data suggested that the merging of Binary Black Holes (BBHs) had a variety of structures in the principal mass function with low spin [6]. Izquierdo-Villalba D estimated the amplitude of the random gravitational wave background produced by unresolved massive black hole binaries at the nanohertz frequency [7]. Chen N established and tested the dynamical friction of massive black hole pairs in bulk cosmological hydrodynamical modelling method. The validation showed that black holes could effectively sink to the centre of galaxies as long as the black hole mass is at least twice the mass of the lowest mass resolution particle [8]. Wang L simulated intermediate-mass globular clusters ($N=100,000$) with primordial binary ratios between 0 and 1 using the high-performance N-body code PETAR. He found that the structural evolution of clusters in the presence of stellar-mass black hole subsystems was mainly determined by massive primordial binaries [9]. Wang D proposed a logarithmic parametrization model of the energy density of scalar-field dark energy, which supported the transition of the Universe from deceleration to acceleration. It was also found that black hole mass growth stops during the dark energy dominated period [10].

Chanda A studied the evolution of the Primordial Black Hole (PBH) under $f(Q)$ -modified gravity, focusing on the accretion of matter from the cosmic fluid around the PBH. The results showed that under $f(Q)$ -modified gravity, the PBH increases in mass at an early stage and eventually saturates, which was different from general relativity and gravity [11]. Gualandris A investigated the change in eccentricity of supermassive black hole binaries from the early stage of the merger through to the hardening phase, and developed a semi-analytical model [12]. Mazzolari G assessed the formation efficiency of extreme mass-ratio inhalation for different massive black hole binary parameters through relativistic three-body simulations, and found that the extreme mass-ratio inhalation formation rate peaks in the sharp outbursts following galaxy mergers, and was a factor of 10-100 higher than the standard relaxation channel [13]. Mapelli M studied the cosmic evolution of binary black holes in four different environments (nuclear clusters, globular clusters, and young clusters) using an upgraded version of the semi-analytic code [14]. Costa G studied the formation of black holes by stellar collisions and the filling of unstable mass gaps in young stellar clusters. Hydrodynamic simulations of collisions between core-burning helium stars and main-sequence stars take into account the mass loss and the post-collision products are modelled with stellar evolution codes [15].

Many advances have been made in the field of black hole dynamical evolution but still face some bottlenecks. Although simulations and observations provide valuable data on black hole growth, mergers, and their relationships with host galaxies, these studies usually rely on specific models and assumptions, limiting a comprehensive understanding of complex physical processes. For example, uncertainties remain about the main pathways of black hole mass growth, the long-term effects of black holes on cluster dynamics, and the formation mechanisms of binary black hole systems in different environments. In addition, current simulations are still limited by computational resources when dealing with large-scale and high-resolution data, which makes it difficult to conduct systematic studies in a wider parameter space.

3. Methods

3.1 Overview of the Development of Numerical Relativity

The theory of relativity abandoned Galileo's and Newton's absolute view of space-time as separate and unrelated, thus achieving a quantum leap in the conception of space-time. The development of the concept of space-time has caused a lot of numerical inconvenience. In practical applications, the physical problem to be studied is often converted into an initial value problem by transforming it into

an equation containing the partial derivatives of each physical quantity with respect to time, thus giving it an initial value at a certain point in time, and then numerically integrating the time evolution of it. As stated before, Einstein field theory is established in four dimensions, and this equation is a covariant tensor equation which can be chosen arbitrarily once, or arbitrarily once in three dimensions, and by choosing different coordinates, the Einstein field theory problem is decomposed into a variety of PDE system (Partial Differential Equation system). On this basis, we transform the numerical solution of Einstein's field theory into two steps: in the first step, in Einstein's field theory, appropriate time coordinates are chosen to partition the four-dimensional space into a series of null-like hypersurfaces; in the second step, the original Einstein's field theory equations are transformed into a set of differential equations containing the time partial derivatives, i.e., the 3+1 solution, and the numerical solution of the Einstein's field theory is decomposed into two steps: firstly, in the conditions of a given value of the initial edge, which is solved numerically; and secondly, on the basis of the given initial value problem, which is solved numerically and then the evolution of each physical quantity in the corresponding time (evolution problem) is obtained.

3.2 General Relativity

Einstein's equations are at the heart of general relativity, bridging the gap between the structure of spacetime (i.e., how matter moves) and the distribution of matter (i.e., how the structure of spacetime arises). Einstein's equations can be derived from Rasch's quantities, and the action quantities describing the whole system can be written as:

$$S = S_g + S_m \quad (1)$$

Where S_g is the action quantity of the gravitational field, also known as the Hilbert action quantity, written in our form as:

$$S_g = \frac{1}{16\pi} \int_M R \sqrt{-g} d^4x \quad (2)$$

Where R is the Ricci scalar and $\int_M$ is the action of all the remaining "matter" fields. We can vary the action by perturbing the inversion gauge to create the Euler-Lagrange equation. After a series of derivations, the variational component of the standard gravitational part of the action can be written as follows:

$$\delta S_g = -\frac{1}{16\pi} \int_M (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \sqrt{-g} d^4x \quad (3)$$

The degree gauge for the material part is written as:

$$\delta S_m = \frac{1}{2} \int_M T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d^4x \quad (4)$$

We define the co-strength-energy tensor density corresponding to the matter part as $\sqrt{-g} T_{\mu\nu}$, which serves as the derivative of the matter action quantity with respect to the inversion gauge.

$$\sqrt{-g} T_{\mu\nu} = 2 \frac{\delta S_m}{\delta g^{\mu\nu}} \quad (5)$$

Since $\delta g^{\mu\nu}$ can be taken arbitrarily, we obtain the Einstein equations describing the highly nonlinear relationship between the degree gauge and the co-strength-energy field:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \quad (6)$$

Where $T_{\mu\nu}$ is the co-strength energy tensor, and $G_{\mu\nu}$ is the Einstein tensor, and $R \equiv R^{\mu\nu}$ is the Ricci scalar, and $R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu}$ is the Ricci tensor.

$$R^{\sigma}_{\mu\rho\nu} \equiv \partial_{\rho} \Gamma^{\sigma}_{\mu\nu} - \partial_{\nu} \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\tau\rho} \Gamma^{\tau}_{\mu\nu} - \Gamma^{\sigma}_{\tau\nu} \Gamma^{\tau}_{\mu\rho} \quad (7)$$

$$\Gamma^{\sigma}_{\mu\rho} \equiv \frac{1}{2} g^{\sigma\tau} (\partial_{\mu} g_{\rho\tau} + \partial_{\rho} g_{\mu\tau} - \partial_{\tau} g_{\mu\rho}) \quad (8)$$

Equation (8) is in Christofel notation and can be expressed in terms of the degree gauge and its first order derivatives. All of the above are four-dimensional quantities defined in four-dimensional spacetime.

For these detectors, double black hole mergers are one of the most important sources of gravitational waves. The energy of gravitational waves in the universe is extremely weak, and theoretical understanding of gravitational waves is limited. These two characteristics make the

detection of gravitational waves challenging. For example, if the waveforms that generate gravitational waves can be correctly identified, matched filter data analysis methods will greatly improve the ability to observe gravitational waves. Currently, numerical computational methods are the only viable theoretical approach that can provide a thorough theoretical explanation of twin black hole mergers. Therefore, numerical computation of Einstein's equations, or numerical relativity, was developed as a separate branch of research. Black holes have a physical spacetime singularity, the point at which the gravitational field (curvature) becomes infinite. It is therefore crucial to choose a computational strategy that avoids these singularities, but this is by no means easy. In addition, regions of strong gravitational fields close to black holes are usually solved for using most of the computational resources; however, since gravitational waves hardly affect the background gravitational field, numerical methods must be able to reach distant regions where the gravitational field is very weak. Numerical relativists also need to determine which form of Einstein's equations is suitable for producing stable numerical solutions, since other forms of the equations behave very differently in numerical calculations.

4. Results and Discussion

4.1 Experimental Preparation

The experimental equipment consists of a high-performance computing cluster and numerical relativistic simulation software for solving the Einstein field equations. The experimental process starts with an initialization setup, including the selection of parameters such as the initial mass, spin and charge of the black hole. Subsequently, numerical simulations of black hole mergers, accretion growth, and other related phenomena are performed using the 3+1 decomposition method.

4.2 Experimental Results

Table 1. Black hole merger dynamics

Initial Mass 1 ($M_{\odot}$)	Initial Mass 2 ($M_{\odot}$)	Spin 1	Spin 2	Peak Amplitude (10^{-21})	Peak Frequency (Hz)	Duration (ms)
10	10	0	0	1.5	150	5
20	20	0.5	0.5	2	200	7
30	30	0.9	0.9	2.8	250	10
50	50	0.5	0	3.1	300	12

In a dynamical experiment of black hole mergers, we explore the effect of different initial masses and spins on the emission of gravitational waves. The data show that the peak amplitude of gravitational waves increases significantly with increasing initial black hole mass, from 1.5×10^{-21} to 3.1×10^{-21} . This suggests that larger mass black holes contribute more significantly to gravitational waves, as shown in Table 1.

Table 2. Growth of black holes through accretion

Initial Mass ($M_{\odot}$)	Accretion Rate ($M_{\odot}/\text{year}$)	Final Mass ($M_{\odot}$)	Accretion Disk Mass ($M_{\odot}$)	Growth Rate ($M_{\odot}/\text{year}$)
50	0.1	60	10	0.1
50	0.5	100	50	0.5
50	1	150	100	1
100	0.5	200	100	0.5

As can be seen from the data in Table 2, the initial mass and acceleration have an impact on the final mass, the stellar disc mass and the growth rate. For example, keeping the initial mass at 50 $M_{\odot}$

$\odot$, the larger the deposition rate, the larger the final mass, the stellar disc mass and the growth rate. For example, when the deposition rate is increased from $0.1 M_{\odot}/\text{year}$ to 1, the final mass, the stellar disc mass and the growth rate increase by $90 M_{\odot}$, $90 M_{\odot}$ and 0.9, respectively.

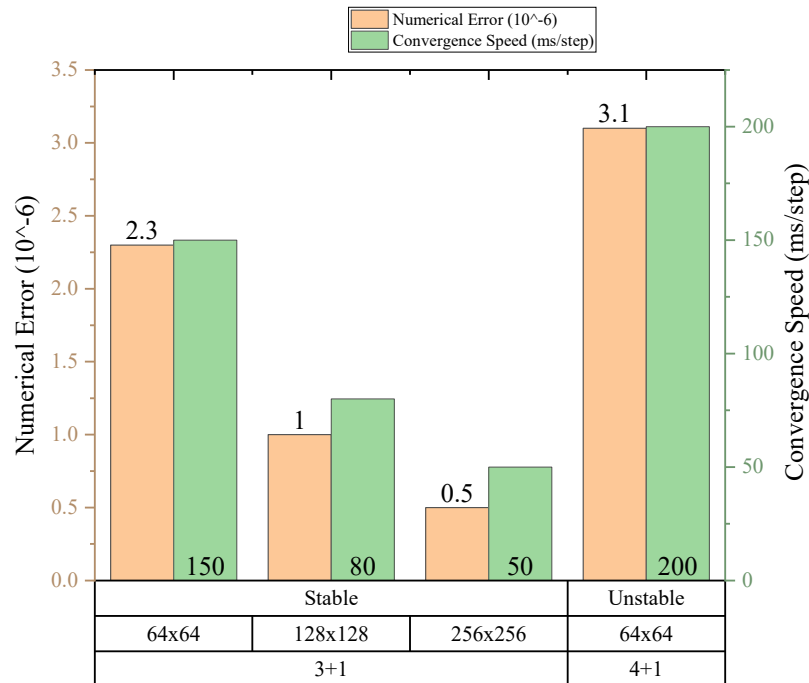


Figure 1. Numerical stability of Einstein's equations

In the experiments on the numerical stability of Einstein's equations, we compared the numerical performances of the 3+1 decomposition and the 4+1 decomposition, which may be more suitable when dealing with complex black hole problems, as it can provide more stable solutions while maintaining lower numerical errors, as shown in **Figure 1**.

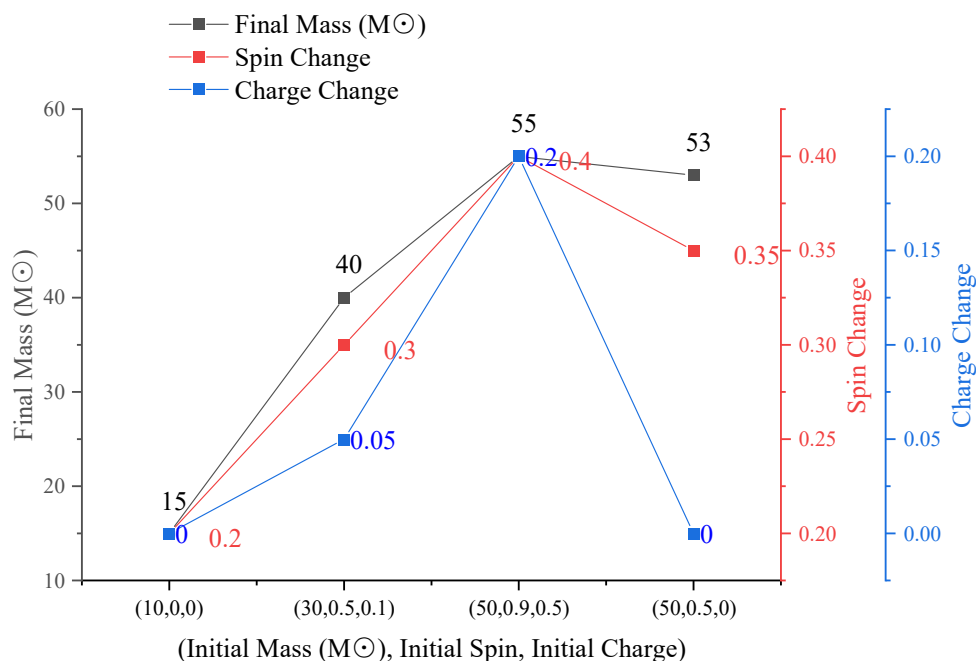


Figure 2. Effect of different initial conditions on the evolution of black holes

In experiments on the effect of different initial conditions on the evolution of black holes, we investigate the effect of black hole mass, spin and charge on their final state. The data show that the initial mass has a significant effect on the final mass, e.g., a black hole with an initial mass of $10 M_{\odot}$ has a final mass of $15 M_{\odot}$, while a black hole with $50 M_{\odot}$ has a final mass of $55 M_{\odot}$. These results suggest that initial conditions have a significant effect on the final mass, spin and charge of a black hole, which is crucial for understanding the evolutionary mechanisms of black holes, as shown in Figure 2.

5. Conclusion

The detection of gravitational waves gives us a relatively independent way to study this mechanism. The time-series signals of gravitational waves produced by the merger of two black holes are able to measure several of their physical parameters (e.g., mass, spin, photometric distance, etc.). With the deepening of gravitational wave detection, the samples of the merger of two black holes will be increasing, so that the physical parameters, external parameters and the correlation nature between each parameter of the two black holes can be obtained. The research results in this paper will help to deeply understand the physical properties of black holes, the most peculiar dense stars, and provide important theoretical basis for their formation and evolution. In this paper, the dynamic evolution of black holes is investigated by numerical relativistic simulations, and the emission characteristics of gravitational waves during black hole merger are thoroughly discussed. The experimental results show that the peak amplitude of gravitational waves can reach 3.1×10^{-21} with a frequency of 300 Hz, and the peak amplitude varies with the initial black hole mass, spin, and accretion rate. These findings significantly enhance the understanding of the behaviour of black holes and their gravitational wave emission conditions, and provide key data for the design and data analysis of gravitational wave detectors. Despite the progress made, there are still some limitations in this study. The simulations are limited by computational resources, resulting in constraints on resolution and parameter space. The assumed initial conditions (e.g., specific mass, spin, and charge values) may not cover the full diversity of black hole systems in the Universe. In addition, complex physical phenomena in the black hole environment (e.g., magnetic field and surrounding matter) are simplified in the models, which may neglect some subtle interactions.

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