

Integral Monotonicity on the Duplex-Plane

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Abstract. Duplex numbers are a generalization of real numbers, which are commutable rings with zero factors generated by two real numbers. In this paper, according to the partial order relationship of the duplex number and the decomposition of the duplex number, the monotonicity of the integral on the duplex-plane is studied, and the monotonic representation of the duplex-integral is obtained. This paper finds two null factors in the definition of the expression of the duplex number, and decompose them on this basis, which is helpful for us to analyze its monotonic nature. The monotonicity of the duplex integral will further promote the development of the theory of duplex analysis.

Keywords: Duplex numbers, Monotonicity, Duplex Integral.

1. Introduction

Duplex numbers also called perplex numbers or hyperbolic numbers, which are having zero factors generated by two real numbers and they are commutable rings. Saini [1] gave the definition of duplex number and many properties of duplex analysis. Arif [2] introduced duplex univalent functions and also discuss the properties of a specific class of univalent functions, the definitions and methods used in the article give this paper a reference for the study of duplex numbers. Balankin [3] studied the set of duplex numbers from the point of view of a duplex module and made use of the partial order defined on the set of duplex numbers, with the help of these notions, some fractal-type sets in the four dimensional space are constructed. Sporn [4] considered the history of the duplex numbers and uses the Pythagorean triples to find the duplex numbers are different from the complex numbers. Chandler [5] explained what the duplex number is from a physical point of view and illustrates the important role of the duplex number system in the study of atoms. Luna-Elizarraras [6] introduced the algebra of duplex numbers as a generalization of the field of complex numbers and described how to define elementary functions in such an algebra as well as their inverse functions.

Tan [7] explained what monotonicity is and describe the areas in which it can be applied. Ozturk [8] explained what is duplex number, also proposed the application of the number system to the study of mathematics. Luna-Elizarraras [9] presented a systematic study of the theory of integration of hyperbolic-valued functions from a new point of view where the notion of partial order defined on the set of duplex numbers and the notion of duplex intervals have been taken into consideration. Starting with the definitions of partition of a duplex interval and the definition of duplex integral, the basic theorems of duplex integration are proved. Chandler [10] elaborated on the important position of duplex numbers in algebraic chemistry, and shows that duplex numbers can help solve the problem of logically deriving multiple rules and irregular structures from the same set of atomic numbers. Huang [11] primarily investigated the Schwarz lemma on the duplex plane and successfully obtain an estimation of the modulus of differentiable functions in perplex plane and discussed its potential applications in complex analysis and geometric function theory. Assis [12] present some properties of mathematical and physical interest in generalized algebras of two, three and four dimensions and gave a new formulation for these algebras, stress some important applications related to the wave equation, and emphasize a novel didactic approach to this subject.

Based on the above research, this paper further explores the properties of the integration on the duplex plane and obtains the monotonicity of the integration on the duplex plane. This will give impetus to the study of the duplex number and will contribute to the study of physical chemistry at the mathematical level.

2. Preliminaries

Now let's consider a new number system P , the duplex numbers, also called the perplex numbers. It is defined as

$$D = \{d = a + bh \mid a, b \in \mathbb{R}, h^2 = 1\} \quad (1)$$

The set D is a commutative ring with the operations of addition and multiplication given as

$$\begin{aligned} d_1 d_2 &= (a_1 + b_1 h)(a_2 + b_2 h) \\ &= (a_1 a_2 + b_1 b_2) + (a_1 b_2 + a_2 b_1) h \end{aligned} \quad (2)$$

For $d_1 = a_1 + b_1 h$ and $d_2 = a_2 + b_2 h$. Then this paper defines

$$e^+ = \frac{1+h}{2} \quad \text{and} \quad e^- = \frac{1-h}{2} \quad (3)$$

And this paper can get

$$e^+ e^- = 0 \quad (4)$$

Thus $\{e^+, e^-\}$ is the idempotent base of D . Besides, there also exists

$$(e^+)^2 = e^+ \quad \text{and} \quad (e^-)^2 = e^- \quad (5)$$

And d can be expressed as

$$d = a + bh = (a+b)\frac{1+h}{2} + (a-b)\frac{1-h}{2} = xe^+ + ye^-(x, y \in \mathbb{R}) \quad (6)$$

For the new coordinates, this paper also has the operations of addition and multiplication. Suppose that $d_1 = x_1 e^+ + y_1 e^-$ and $d_2 = x_2 e^+ + y_2 e^-$, then this paper can get

$$\begin{aligned} d_1 + d_2 &= (x_1 + x_2)e^+ + (y_1 + y_2)e^- \\ d_1 d_2 &= (x_1 x_2)e^+ + (y_1 y_2)e^- \end{aligned} \quad (7)$$

Let

$$D^+ = \{d = xe^+ + ye^-; x \geq 0, y \geq 0\} \quad (8)$$

Which called the set of non-negative duplex numbers. In the same way,

$$D^- = \{d = xe^+ + ye^-; x \leq 0, y \leq 0\} \quad (9)$$

Is called the set of non-positive duplex numbers. The image of them is just as Figure 1.

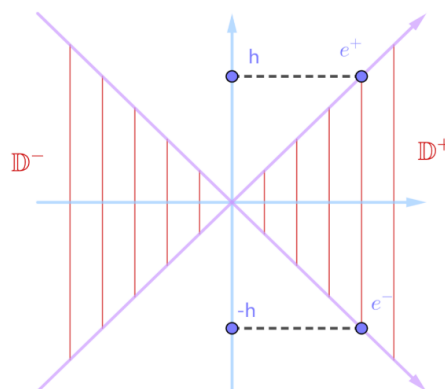


Figure 1. The positive and negative duplex numbers

Obviously if $d_1, d_2 \in D^+$ then $d_1 d_2 \in D^+$, so D^+ is closed under the multiplication.

For $d_1, d_2 \in D^+$, this paper defines

$$d_1 \succeq d_2 \text{ if and only if } d_1 - d_2 \in D^+ \quad (10)$$

And this paper says that d_1 is D -greater than d_2 , or

$$d_1 \preceq d_2 \text{ if and only if } d_1 - d_2 \in D^- \quad (11)$$

And this paper says that d_1 is D -less than d_2 .

Due to the introduction of partial order, this paper can define duplex intervals. The duplex interval is defined as follows:

$$[d_1, d_2]_D = \{d \in D : d_1 \preceq d \preceq d_2\}$$

The image of this interval is Figure 2.

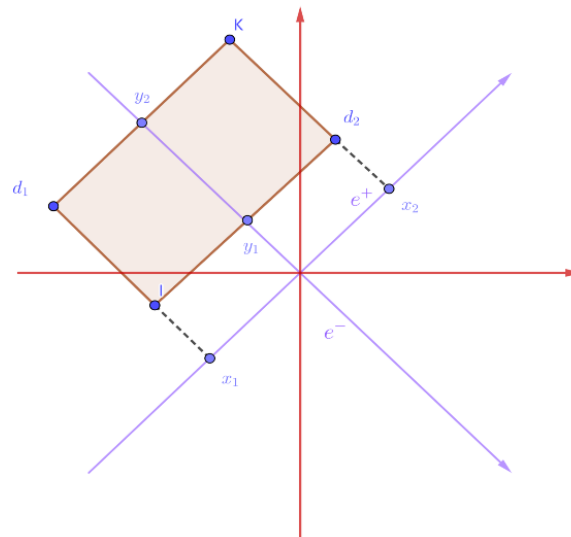


Figure 2. The duplex interval $[d_1, d_2]_D$

Suppose f are functions of a P complex variable:

$$f : U \subseteq D \rightarrow D$$

Here U is an open subset of D . After assuming that $f_1, f_2 \in C(U)$, these functions can be written as $f(d) = f_1(a, b) + hf_2(a, b)$. This paper calls

$$\lim_{t \rightarrow 0} \frac{f(d_0 + t) - f(d_0)}{t} \quad (t \text{ is not a zero factor}) \quad (12)$$

Is differentiable.

Definition: Let f be a d valued function of a d variable. This paper says f is $Cl_{0,1}$ -differentiable if

$$f = f_1(x)e^+ + f_2(y)e^- \quad (1)$$

And f_1, f_2 are real differentiable functions.

Lemma: If $f_1 : A_1 \rightarrow R$ and $f_2 : A_2 \rightarrow R$ are $Cl_{0,1}$ -differentiable on the interval $[A_1, A_2]$ respectively, then this paper can define a function $f : A_1 \times A_2 \rightarrow D$ as

$$f(d) = f_1(x)e^+ + f_2(y)e^- \quad (14)$$

Where $d = xe^+ + ye^- \Rightarrow d(xe^+ + ye^-) = (dx)e^+ + (dy)e^- \quad \forall \quad d \in A_1 \times A_2 \subseteq D$. Thus this paper has

$$\int_H f(d) \otimes d(xe^+ + ye^-) = \left\{ \int_{\gamma_1} f_1(x) \otimes dx \right\} e^+ + \left\{ \int_{\gamma_2} f_2(y) \otimes dy \right\} e^- \quad (2)$$

Where $H = \gamma_1 e^+ + \gamma_2 e^-$.

3. The monotonicity of the duplex integral

The duplex integral plays a very important role in the theory of duplex analysis. In this chapter, the decomposability of $Cl_{0,1}$ -differentiable functions is obtained, and the monotonicity of duplex integrals is obtained based on the partial order properties of duplex complex variable functions. This will lay the foundation for the application of duplex analysis in physics, chemistry, etc.

Theorem (The monotonicity of the duplex integral): The duplex complex valued function f is $Cl_{0,1}$ -differentiable. If $d_1 \preceq d_2$, f and g are $Cl_{0,1}$ -differentiable, and $f(d) \preceq g(d)$ at each point $d \in [d_1, d_2]_D$, then

$$\int_{[d_1, d_2]_D} f(d) \otimes d(xe^+ + ye^-) \preceq \int_{[d_1, d_2]_D} g(d) \otimes d(xe^+ + ye^-) \quad (16)$$

This paper uses Figure 3 to represent it.

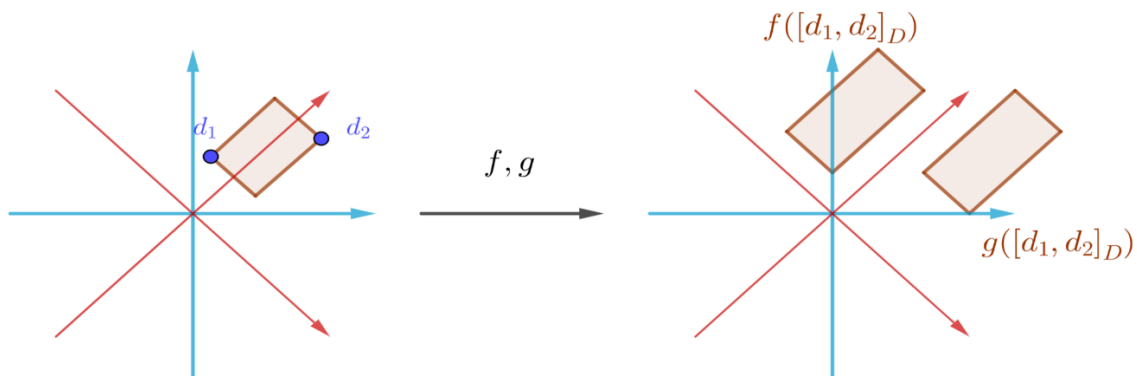


Figure 3. The geometric meaning of monotonicity

Proof:

Let $d_1 = x_1 e^+ + y_1 e^-$ and $d_2 = x_2 e^+ + y_2 e^-$, then the left part of the inequality can be written as

$$\int_{[d_1, d_2]_D} f(d) \otimes d(xe^+ + ye^-) = e^+ \int_{x_1}^{x_2} f_1(x) dx + e^- \int_{y_1}^{y_2} f_2(y) dy$$

According to the monotonicity, this paper can get

$$\begin{aligned} f(d) \preceq g(d) &\Rightarrow f_1(x)e^+ + f_2(y)e^- \preceq g_1(x)e^+ + g_2(y)e^- \\ &\Rightarrow f_1(x) \preceq g_1(x), f_2(y) \preceq g_2(y) \end{aligned}$$

For $\forall x \in (x_1, x_2), y \in (y_1, y_2)$.

Then

$$\begin{aligned}
 e^+ \int_{x_1}^{x_2} f_1(x) dx + e^- \int_{y_1}^{y_2} f_2(y) dy &\preceq e^+ \int_{x_1}^{x_2} g_1(x) dx + e^- \int_{y_1}^{y_2} g_2(y) dy \\
 &= \int_{x_1}^{x_2} g_1(x) d(xe^+) + \int_{y_1}^{y_2} g_2(y) d(ye^-) \\
 &= \int_{[d_1, d_2]_D} (g_1(x)e^+ + g_2(y)e^-) \otimes d(xe^+ + ye^-) \\
 &= \int_{[d_1, d_2]_D} g(d) \otimes d(xe^+ + ye^-)
 \end{aligned}$$

Then this paper can proof that

$$\int_{[d_1, d_2]_D} f(d) \otimes d(xe^+ + ye^-) \preceq \int_{[d_1, d_2]_D} g(d) \otimes d(xe^+ + ye^-)$$

4. Conclusion

The Duplex number is similar to the complex algebraic form, but the structure is different. The Duplex number is an exchangeable ring with a zero factor generated by two real numbers. In this paper, the integral theory on the Duplex-plane is studied, and the relationship between the partial order of functions and the partial order of integral functions is obtained, which provides a theoretical basis for the development of physics.

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