Accelerating Expansion of the Universe in the View of Relativity

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Abstract. The 2011 Nobel Prize in Physics was granted to Saul Perlmutter, Brian Schmidt, and Adam Reese, who found that the universe is accelerating its expansion and overturned the traditional view. This paper discusses the expansion of the universe based on the theory of general relativity. Some basic principles of the relativity theory are introduced first, which draws the problem of universe evolution. Then we explore the secrets of the universe's accelerating expansion with several basic formulas from the special theory of relativity and the general theory of relativity.

Keywords: Special Relativity; General Relativity; Universe Expansion; Friedman Universe.

1. Principle of Special Relativity

Newton's great work "Mathematical Principles of Natural Philosophy", published in 1687, marks the birth of modern physics and builds a nearly perfect mechanical system in logic. In Newtonian mechanics, the inertial system is defined as a reference system that is stationary relative to the absolute space or moves at a constant speed in a straight line. The laws of mechanics are valid in all inertial reference systems and meet the principle of relativity: all inertial systems are equal in describing the laws of mechanics. However, with the establishment of the electromagnetic theory, people urgently realize that electromagnetic waves need a stationary carrier in the existing mechanical framework system. But the famous Michelson experiment and Fizeau experiment deny the existence of this static carrier. At the same time, Maxwell's electromagnetic theory concludes that the propagation speed of the electromagnetic wave in vacuum is a constant. Then the question arises: if there is no absolute stationary reference system, which reference system is the speed of light defined relative to?

Einstein grasped the main contradiction and proposed the principle that the speed of light in vacuum is a constant in all inertial reference frames. And he also realized the importance of the principle of relativity, which is more essential than the Galilean transformation of straight inertial space-time in the past. Based on the invariance nature of the light speed and the principle of relativity, he created a new field different from classical mechanics: special relativity. Lorentz transformation is used in Einstein's theory, which is different from Galileo's transformation of classical mechanics. Next, we briefly introduce a method to derive Lorentz transformation from the space-time interval invariant. Because of the equivalence of time and space in relativity, The event has the invariance of space-time interval. The space-time interval is defined as follows:

$$-c^2\Delta\tau^2=-c^2\Delta t^2+\Delta x^2+\Delta y^2+\Delta z^2 \tag{1}$$

where c is the light speed, x, y, and z are spatial coordinates, and t is the time coordinate.

In the special theory of relativity, we discuss the four-dimensional Minkowski flat spacetime. For the convenience of discussions, we introduce some mathematical expressions commonly used in relativity. We use a general four-dimensional vector (i.e., first-order tensor) x_α to describe the coordinates of flat space-time,

$$x_{1,2,3,4}=(x, y, z, \mathrm{i}ct) \tag{2}$$

In this form, our spacetime interval ds^2 can be expressed in the form of the multiplication of two vectors

$$ds^2=dx_\alpha dx_\alpha \tag{3}$$

Suppose that there are two inertial reference systems, the S reference system, and S' reference system, in which S' reference system moves in a straight line with a speed of v in the x direction at a constant speed relative to the S reference system, then we consider a clock bound to a moving particle (that is, the displacement between two events is 0), then the $\delta\tau$ is defined as the fixed time in relativity. Since the event displacement in S' is 0, the observation in the S system tells us the displacement of the object $\Delta s = v\Delta t$, By introducing this formula into Eq.(1), we can get the conversion relationship between solid time and time in the S system

$$\Delta\tau=\sqrt{1\text{-}v^2/c^2}\Delta t \tag{4}$$

That is to say, for people in the S series, the interval between two events will be less than the fixed time and become a little slower, which is also known as the clock slow effect. In relativity, $\gamma=1/\sqrt{1\text{-}v^2/c^2}$ is called Lorentz factor. We define the form of Lorentz transformation under the above two reference systems as:

$$dx_\mu=a_{\mu\alpha}dx_\alpha,\mu,\alpha=1,2,3,4 \tag{5}$$

where $a_{\mu\alpha}$ is a transformation matrix, which is a linear transformation due to the characteristics of flat space-time. As mentioned above, the principle of special relativity requires that the space interval remains unchanged under Lorentz transformation, so we can get from Eq. (3):

$$dx_\alpha dx_\alpha=dx_\mu dx_\mu \tag{6}$$

This requires the matrix $a_{\mu\alpha}$ Is an orthogonal matrix. That is, transpose matrix equals inverse matrix. Using undetermined coefficient method, we get:

$$a=\begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \text{-}i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{7}$$

Substituting it to Eq. (5), we can get the Lorentz transformation form:

$$\begin{cases} x' = \gamma\left(x - vt\right) \\ t' = \gamma\left(t - \dfrac{vx}{c^2}\right) \\ y' = y \\ z' = z \end{cases} \tag{8}$$

The laws of kinematics and electromagnetism satisfy the four-dimensional space-time tensor equation under the special theory of relativity (i.e., it is consistent with Lorentz covariant form). The space-time interval we derived above is the most classical one satisfying the Lorentz covariant.

2. Principle of General Relativity

Special relativity is based on the inertial system. If there is no absolute reference system, the inertial system cannot be strictly defined. If there is nothing in space, we cannot distinguish and mark various motions at all, let alone define the motional state of a static or uniform straight line. Special relativity can perfectly conform to the laws of electromagnetism and dynamics, but it is not compatible with the law of universal gravitation. In this case, the law of universal gravitation in the form of Lorentz covariant cannot be established. So this theory lacks a key. Einstein grasped the principle of equivalence, which is a lifesaving straw. Since the inertial reference system cannot be defined, it is better to extend this inertial reference system to any reference system. This means that the basic principles of special relativity: the principle of invariance of the speed of light and the principle of relativity are valid for any reference system. Then we have the general relativity principle: equal rights of all reference systems, The form of physical laws remains unchanged in any coordinate system. The emergence of the principle of general relativity naturally requires the existence of inertial forces in our previous noninertial frames, to add new effects to the system to meet the generalized covariance. Inertial force does not originate from the interaction between substances, and there is no reaction force. Mach believed that the inertial force originated from the accelerated motion of the force-bearing object relative to the distant galaxy, which has a similar origin to gravity. At present, contemporary physicists are still constantly exploring the generation mode of inertial force. Inertial force is proportional to the mass of matter, and this is similar to Newton's law of universal gravitation that we have learned, which enables the mass of matter to be defined by two different forces: inertial mass and gravitational mass. The gravitational mass and inertial mass are completely consistent under the current experimental accuracy. But no one has yet concluded whether the principle of equivalence (equivalence of gravitational field and inertial field) is true. The Lorentz transformation, the basis of the special theory of relativity that we derived above, has become a special case form of the inertial system in the general theory of relativity. To extend it to all reference systems, the four-dimensional flat spacetime is no longer applicable. For example, in a non-flat space, the transformation matrix above is no longer a simple linear orthogonal matrix. Similarly, we can start from the time-space interval, but to meet the requirement that the quantity is a scalar (i.e., invariant in affine space), we need to introduce the concept of metric based on Eq. (3) to meet the covariance of the equation. That is, Eq. (3) becomes:

$$ds^2=g_{\mu\nu}dx_\mu dx_\nu \tag{9}$$

where $g_{\mu\nu}$ is called a metric tensor, which is a second-order covariant tensor in mathematics. With the concepts of tensor and metric, Einstein was finally able to establish the core of general relativity: Einstein's field equation. To satisfy the generalized covariance, Einstein guessed that the equation was a tensor equation, which should conform to the principle of equivalence, the principle of invariance of the speed of light, and the geodesic equation (which can be simply understood as the Riemannian geometric version of Newton's first law). In addition, he combined Mach's guess to attribute the interaction between distant galaxies and accelerating objects to the curvature of space-time and wrote the field equation in a form similar to the Poisson equation we have learned.

$$G_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu} \tag{10}$$

where $G_{\mu\nu}$ is the famous Einstein tensor. Its expression contains curvature tensor and metric tensor, describing how space-time bends at each point. Matter in space-time defines another quantity: energy-momentum tensor $T_{\mu\nu}$. This field equation nicely explains the principle of "space-time tells matter how to move, and matter tells space-time how to bend" under the theory of general relativity.

3. Friedman Universe in General Relativity and its Expansion Rate

We briefly introduced general relativity and Einstein's field equations in the previous chapter. Now we can establish the standard dynamic model of the evolution of the universe - the Friedman universe. In recent years, a large number of astronomical observations have found that the distribution of galaxies and galaxy clusters is isotropic, and Copernicus's heliocentric theory also leads to the advanced version of the theory that the universe has no center, both of which lead to the principles of modern cosmology, that is, the three-dimensional space of the universe is isotropic at any time, which means that the curvature of three-dimensional space should be the same everywhere on the cosmic scale. In combination with general relativity, a new metric called Robertson Walker metric is introduced, and its four-dimensional general form is:

$$ds^2\text{=-}dt^2\text{+}a^2(t)\left[\frac{dr^2}{1\text{-}Kr^2}\text{+}r^2\left(d\theta^2\text{+}\sin^2\theta d\phi^2\right)\right] \tag{11}$$

where K is a function that only depends on time, and $a(t)$ is a cosmic scaling factor, which is also a function that only depends on time and is greater than 0. This metric is the core of modern kinematic cosmology. The energy-momentum tensor of fluid conforming to this metric can be written in the form of density pressure, and then brought into Einstein field Eq. (10), we can get the basic formula in cosmology - Friedman equation (for the sake of simplicity, the natural unit system c=1 is used here), and its time component equation is:

$$3\frac{da^2}{dt^2}\text{=-}4\pi G(\rho\text{+}3p)a \tag{12}$$

The spatial component equation is

$$\mathrm{a}\frac{da^2}{dt^2}\text{+}2\left(\frac{da}{dt}\right)^2\text{+}2K\text{=}4\pi G(\rho) \tag{13}$$

The cosmic expansion rate H to be discussed is defined as the first-order time derivative of the cosmic scale factor divided by the cosmic scale factor.

$$H\text{=}\frac{1}{a}\frac{da}{dt} \tag{14}$$

It is worth mentioning that the expansion rate H of the universe here is expressed in capital H because its definition is related to the Hubble constant in the famous Hubble Law. Hubble's Law describes the direct relationship between the retrogression speed of distant galaxies and the distance from the Earth.

$$v\text{=}H_0D \tag{15}$$

H_0, also known as the Hubble constant, can be measured by the la type supernova standard candlelight method, cosmic microwave background radiation method, and other methods. It reflects the expansion rate of the universe today. Generally, the measured value is around 65-75 (km/s)/Mpc, and the current Hubble constant is larger than the value observed billions of years ago, which indicates that the expansion of the universe today is accelerating.

Let's go back to the Friedman equation and bring Eq. (14) back to Eq. (12) and Eq. (13) to get two equations about the expansion rate of the universe.

$$H^2\text{=}\left(\frac{1}{a}\frac{da}{dt}\right)^2\text{=}\frac{8\pi G}{3\rho}\text{-}\frac{k}{a^2} \tag{16}$$

$$\frac{dH}{dt}\text{+}H^2\text{=}\frac{1}{a}\frac{da^2}{dt^2}\text{=-}\frac{4\pi G}{3}(\rho\text{+}3p) \tag{17}$$

With the above two formulas, we can simply analyze the expansion rate of the universe. From Eq. (17), we can find that if the universe is full of baryonic matter that we are familiar with, then there will be a strong energy condition for barbarians $\rho + 3p>0$, then we will get $\ddot{a}<0$, $a>0$. Combined with the current measured Hubble constant mentioned above $da/d\rho>0$, then the universe should undoubtedly be in the stage of deceleration expansion, which is inconsistent with the observation fact that the universe is in accelerating expansion. In this case, to reverse this situation, matter with negative pressure must exist, or we need to modify Einstein's field equation. Astronomical circles place their hopes on dark energy, the fuel that drives the universe's accelerating expansion. We will discuss it further below.

4. Cosmological Constants and Dark Energy of Dark Matter

In the preceding content, we mentioned the conflict between the accelerated expansion of the universe and the strong energy condition of Einstein's field equation. To solve this problem, the cosmological constant Einstein once tried to add has returned to the vision of astrophysicists. The cosmological constant first appeared in Einstein's paper entitled "Cosmological considerations in general relativity" in 1917. In this paper, he tried to add the cosmological constant term to Einstein's tensor, because a static universe recognized at that time needed the constant to counter gravity. After adding the cosmological constant, Friedman equation (17) became:

$$\frac{dH}{dt}+H^2=\frac{1}{a}\frac{da}{dt}=-\frac{4\pi G}{3}(\rho+3p)+\frac{\Lambda}{3} \tag{18}$$

Einstein tried to use this constant to build a finite universe that is stable to gravitational collapse, balancing out the right term of the above formula. But when Hubble observed an expanding universe, Einstein completely gave up the cosmological constant and claimed that the introduction of cosmological terminology was the biggest mistake in his life. However, it is mentioned above that modern astronomical observations support an accelerating expanding universe, and a positive cosmological constant term can naturally solve the problems mentioned in the previous chapter. As for what is cosmological constant is, with the development of quantum mechanics, physicists were surprised to find that the quantum field theory indicates that a large portion of the energy is hidden in the vacuum, and it has the same properties as a cosmological constant when it is brought into Einstein's field equation as an ideal fluid! However, this does not mean that vacuum energy is the answer to the question. The vacuum energy calculated by modern quantum mechanics is far greater than the cosmological constant, which means that the current theory still cannot explain the cosmological constant term. So the concept of dark energy, a big fire, was put forward, but it is far from the fantasy people thought. It is just a form of energy that fills space and increases the expansion speed of the universe. In addition to acting as cosmological constant terms, many emerging studies now also believe that dark energy may be the fifth fundamental force in nature, and there is a new scalar field that can independently drive the universe to accelerate expansion. Scalar field theory allows dark energy to have a certain degree of inhomogeneity. To avoid too much non-uniformity, the mass of this scalar field must be very light, so that a large Compton wavelength can be generated to avoid detection in our ground laboratory and play a role in the cosmological scale.

In addition to dark energy, what role does dark matter play in the evolution of the universe? Dark matter is proposed by astronomers to solve the problem that the rotation curve of galaxies is completely inconsistent with Newton's gravitational prediction, which means that a material that does not interact with electromagnetic force is needed, that is, a material that does not absorb, reflect or emit light but has a gravitational effect. Later, with the development of astronomy and particle physics, cosmic microwave background radiometry also supports the existence of dark matter, and particle physicists are constantly looking for particle candidates corresponding to dark matter.

Dark matter is the opposite of dark energy. It has the same gravitational effect as baryonic matter, which means that it does not drive the universe to expand faster like dark energy, but restricts the expansion of the universe in reverse. If we use density parameter to express Friedman equation 16 with cosmological constant added, we will get

$$H^2/H_0^2=\Omega_R a^{-4}+\Omega_M a^{-3}+\Omega_k a^{-2}+\Omega_\Lambda \tag{19}$$

where Ω_M represents the current density of matter in the universe including baryons and dark matter. If the cosmological constant is equivalent to the dark energy Ω_Λ, it represents the density of dark energy. It can be seen that these two terms together affect the expansion rate of the universe.

References

[1] Ferreira, P. G. (2019). Cosmological Tests of Gravity. Annual Review of Astronomy and Astrophysics, 57, 335–374.

[2] Peebles, P. J. E. (2012). Seeing cosmology grow. Annual Review of Astronomy and Astrophysics, 50, 1–28.

[3] Joyce, A., Lombriser, L., & Schmidt, F. (2016). Dark Energy Versus Modified Gravity. Annual Review of Nuclear and Particle Science, 66, 95–122.