

The solution of inverse matrix by using unary polynomial ring theory

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Abstract. Matrix is in the core position in algebra, and the inverse matrix is particularly important. So far, there are adjoint matrix method, elementary transformation method, block matrix method and so on in solving the inverse matrix. In this paper, we will introduce a method of seeking the inverse matrix by using unary polynomial ring theory. This method can reverse the matrix equation into the algebra equation, cleverly avoiding the complex matrix equation.

Keywords: inverse matrix, unary polynomial ring theory

1. Introduction

There are not unique methods to solve the inverse matrix. However, different methods mean that we have to face different amount of calculation. And the appropriate method can let us avoid unnecessary complex calculation. According to the current research, Zhang Shenggui use Grobner's basis theory to find the minimal polynomial of matrix, determine the invertibility of matrix and find the inverse matrix [1]. He also gave the necessary and sufficient conditions for the invertibility of cyclic matrices in the domain and two algorithms for finding the inverse matrix. By studying abstract matrices, Xue Weishun and Li Xiulan summarized a variety of methods to find the inverse matrix, and simplified the problem of finding the inverse matrix [2]. Ma Xueling and Zhan Jianming used Hamilton-Cayley theorem and Gauss-Jordan theorem to find inverse matrix [3]. Through the study of symmetric matrices, Tian Jinling obtained the inverse matrix of symmetric cyclic matrix and full symmetric matrix by the method of congruent transformation and block matrix [4]. Chen Fei and Liu Huaiwang gave three methods for general matrix equation when the coefficient matrix satisfies different conditions [5]. At the same time, the algorithm steps and calculation examples are given. However, they have not made a detailed theoretical explanation for solving the inverse matrix by unary polynomial ring. Therefore, in this paper, inverse matrix is solved by a polynomial ring of one variable. Then the algebraic multiplication of the two expressions is done and transformed into an equation problem. Finally, the inverse matrix can be obtained by using the known matrix to express the unknown parameters in the equation, and an example is given.

2. Unary polynomial ring

2.1. Definition of polynomial

Definition 1:

K is a number field and x is an undetermined element. The polynomial is like the following expression

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0, \quad (a_i \in K, 0 \leq i \leq n) \quad (1)$$

Then $f(x)$ become a unary polynomial with x as an undetermined element.

2.2. Addition of polynomial

Definition 2:

Addition of polynomials, if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0, \quad (a_i \in K, 0 \leq i \leq n) \quad (2)$$

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x^1 + b_0, \quad (b_i \in K, 0 \leq i \leq m) \quad (3)$$

Assuming that $n > m$, then let that

$$f(x) + g(x) = (a_n)x^n + \dots + (a_{m+1})x^{m+1} + (a_m + b_m)x^m + \dots + (a_1 + b_1)x^1 + a_0 + b_0, \quad (a_i, b_i \in K, 0 \leq i \leq n) \quad (4)$$

2.3. Multiplication of polynomial

Definition 3:

Multiplication of polynomials, If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0, \quad (n \neq 0) \quad (5)$$

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x^1 + b_0, \quad (m \neq 0) \quad (6)$$

then let that $h(x)$ is the product of these two polynomials and the degree of $h(x)$ is $m+n$. So

$$h(x) = C_{m+n} x^{m+n} + \dots + C_1 x^1 + a_0 b_0, \quad (7)$$

among:

$$C_{m+n} = a_n b_m, \dots, C_k = \sum_{i+j=k} a_i b_j, \dots, C_0 = a_0 b_0. \quad (8)$$

2.4. Properties of polynomial ring

Theorem 1:

If a polynomial in the number field K satisfies three definitions, and the multiplication of polynomials also satisfies the following properties

(1) Commutative law:

$$f(x) \cdot g(x) = g(x) \cdot f(x) \quad (9)$$

(2) Associative law:

$$(f(x) \cdot g(x)) \cdot m(x) = f(x) \cdot (g(x) \cdot m(x)) \quad (10)$$

(3) Allocation rate:

$$(f(x) + g(x)) \cdot m(x) = f(x) \cdot m(x) + g(x) \cdot m(x) \quad (11)$$

(4) Multiplicative identity element:

$$1 \cdot f(x) = f(x); \quad (12)$$

(5) Number multiplication and multiplication exchange law:

$$(kf(x))g(x) = f(x)(kg(x)) = (f(x)g(x))k \quad (13)$$

then $K[x]$ is a ring of unary polynomials in the number field K .

3. Inverse matrix

3.1. The definition of the inverse matrix

Definition 4:

First of all, for any square matrix A of order n we have $AE=EA=A$ where E is the identity matrix of order n .

Let A be a matrix of order n . If there exists another matrix B of order n such that

$$AB = BA = E \quad (14)$$

then the square matrix A is invertible and the square matrix B is the inverse of A .

3.2. Related properties of inverse matrices[6-13].

(1) The status of A and B is equal, so A and B are inverse matrices of each other, also called A is the inverse matrix of B ;

(2) The identity matrix E is invertible. Namely: $E=E^{-1}=E$;

(3) The null matrix is not invertible, that is, you can not get B , such that: $OB=BO=E$;

(4) If A is invertible, then A inverse is unique;

(5) If A is invertible, then A minus 1 is invertible, $A=(A^{-1})^{-1}=A$;

(6) If A is invertible, transposition of A is invertible. Namely: $(A^{-1})^T=(A^T)^{-1}$;

(7) If A and B are square matrices of the same order and are both invertible, then AB is invertible: $(AB)^{-1}=A^{-1}B^{-1}$.

4. The Method of Inverse Matrix

4.1. General properties of polynomial ring $K[x]$

Definition 5:

Let the set R be a ring and U a subset of R . If U is a nontrivial subring of R , and R is called an extension ring of U .

According to the complete square formula:

$$\begin{aligned}(x+a)^2 &= (x+a)(x+a) \\ &= x^2 + ax + xa + a^2 \\ &= x^2 + 2ax + a^2\end{aligned}\tag{15}$$

So, let A be a matrix of order n in the number field K . By using the allocation law of matrix multiplication, we can obtain:

$$\begin{aligned}(A+aE)^2 &= (A+aE)(A+aE) \\ &= A^2 + A(aE) + (aE)A + (aE)(aE) \\ &= A^2 + 2aA + a^2E\end{aligned}\tag{16}$$

it follows that x can be completely replaced by the matrix A in the complete square formula. And by the same token, we can also use x instead of A in the complete square of A matrix. From the general properties of the polynomial ring $K[x]$, we can know the matrix A is directly replaced by x , and the identity matrix E is replaced by 1. In the polynomial of K , the indefinite element x can be substituted by any element in an extension ring R of K .

4.2. Make assumptions

Let A be an n -order identity matrix. In addition, it is assumed that the inverse of matrix A is a polynomial of matrix A . If the coefficients of polynomial can be obtained, the assumption is right.

4.3. Verify assumptions

Suppose the matrix is T , and $T = (A + cE)$, $A^2 = A$. According to properties, A is replaced by x , and E is replaced by 1. Then assume that $(a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A^1 + a_0 E)$ is the inverse matrix of matrix T , then there will be an equation:

according to conditions $A=A^2 \Leftrightarrow x^2=x$

$$(A + cE)(a_1 A + a_0 E) = E\tag{17}$$

$$\Leftrightarrow (x + c)(a_1x + a_0) = 1 \quad (18)$$

$$\Leftrightarrow a_1x^2 + (a_0 + ca_1)x + ca_0 = 1 \quad (19)$$

$$\Leftrightarrow (a_1 + a_0 + ca_1)x + ca_0 = 1 \quad (20)$$

according to the equality of the corresponding terms on both sides of the equation, it can be obtained that

$$\begin{cases} ca_0 = 1 \\ a_1 + a_0 + ca_1 = 0 \end{cases} \Rightarrow \begin{cases} a_0 = \frac{1}{c} \\ a_1 = -\frac{1}{c(1+c)} \end{cases} \quad (21)$$

the obtained parameters are substituted to obtain the inverse matrix :

$$T^{-1} = -\frac{1}{c(1+c)}A + \frac{1}{c}E \quad (22)$$

thus, the unknowns a_0 and t are solved. So this equation has been proved to have a solution, that is, an inverse matrix can be expressed by a polynomial ring. So, the assumption is right.

4.4. Applied to examples

Example1:

Let A be a 3-order matrix, then $A=A^2$: $(A + 3E)^{-1} = \underline{\hspace{2cm}}$.

Solution: Substitute the number matrix A and the identity matrix E in the univariate polynomial extension ring:

$$(A + 3E) \Leftrightarrow (x + 3) \quad (23)$$

since the inverse matrix formula:

$$(x + 3)(x + 3)^{-1} = 1 \quad (24)$$

so assume:

$$(x + 3)^{-1} = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x^1 + a_0 \quad (25)$$

because of:

$$A^2 = A \Leftrightarrow X^2 = X \quad (26)$$

so:

$$x^i = x(i = 1, 2, 3, \dots, n) \quad (27)$$

so:

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x^1 + a_0 = x(a_n + a_{n-1} + \dots + a_1) \quad (28)$$

suppose r is a constant and:

$$r = a_n + a_{n-1} + \dots + a_1 \quad (29)$$

get the equation:

$$(x + 3)(rx + a_0) = rx^2 + a_0x + 3rx + 3a_0 = x(4r + a_0) + 3a_0 = 1 \quad (30)$$

then the equations are obtained:

$$\begin{cases} 4r + a_0 = 0 \\ 3a_0 = 1 \end{cases} \Rightarrow a_0 = \frac{1}{3}; r = \frac{1}{12} \quad (31)$$

results there:

$$(A + 3E)^{-1} = \frac{A-4E}{-12} \quad (32)$$

Example2:

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -6 & 7 \end{pmatrix}, B = (E + A)^{-1}(E - A), \text{ then } (E + B)^{-1} = \text{_____}. \quad (33)$$

Solution: Replace the number matrices A, B and the identity matrix E in the univariate polynomial extension ring:

$$\begin{aligned} B = (E + A)^{-1}(E - A) &\Leftrightarrow y = (1 + x)^{-1}(1 - x) = \frac{1 - x}{1 + x} \\ (E + B)^{-1} &\Leftrightarrow (1 + y)^{-1} = \frac{1}{1 + y} \end{aligned} \quad (34)$$

derivation:

$$\frac{1}{1 + y} = \frac{1}{1 + \frac{1 - x}{1 + x}} = \frac{1}{\frac{2}{1 + x}} = \frac{1 + x}{2} \quad (35)$$

so:

$$(E + B)^{-1} = \frac{E + A}{2} \quad (36)$$

results there:

$$\begin{aligned} (E + A)^{-1} &= \frac{1}{2} \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -6 & 7 \end{pmatrix} \right] \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -6 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -2 & 1/2 & 0 \\ 0 & 0 & -3 & 4 \end{pmatrix} \end{aligned} \quad (37)$$

Example3:

According to the Properties of Two Polynomials. If $(f(x), g(x)) = 1$, then $u(x)$ and $v(x)$ exist, such that $f(x)u(x) + g(x)v(x) = 1$. where $u(x)$ and $v(x)$ can be obtained by the division method. For example, let $A^3 - A + E = 0$, and solve the inverse matrix of $A^2 + A - E$.

Solution: According to rolling phase division method
so

$$(A^3 - A + E)(A + E) + (A^2 + A - E)(-A^2) = E \quad (38)$$

moreover

$$A^3 - A + E = 0 \quad (39)$$

so

$$(A^2 + A - E)(-A^2) = E \quad (40)$$

results there:

$$(A^2 + A - E) - 1 = -A^2 \quad (41)$$

5. Conclusion

In order to reduce the calculation complexity of the inverse matrix, we propose a method of the inverse matrix by using the theory of polynomial ring. The results show that the matrix A can be

replaced by x and the identity matrix E can be replaced by 1 in the extended ring. The inverse matrix question is transformed into the problem of solving the algebra equations.

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