A Natural Exposition and Concise Derivation of Geometric Optics Laws by Fermat Principle

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Abstract. With Fermat Principle, that is, the most fundamental principle in geometric optics, it is easy to obtain the three laws of geometric optics as well as the paraxial refraction and catoptric imaging laws. Making use of the principle and a simple mathematical relation, this paper expounds and derives the rectilinear propagation law of rays in the homogeneous medium, the optical reflection and refraction laws as well as the spherical reflection and refraction laws under the paraxial condition. Ulteriorly, with given character rules, the spherical dioptric and catoptric imaging formulas can be generalized into a unified formula separately. The elaboration and derivation in this paper reflect the simplicity of physics and reduce the error probability of complex trigonometric function operation, thus making it easier for high school students or junior undergraduate and junior college students to accept and understand.

Keywords: fermat principle, geometric optics law, formula derivation.

1. Introduction

Fermat Principle is also known as the shortest optical path principle[1] that can be specifically expressed as: the actual optical distance between two points P and Q is a smooth path of the optical path (PQ), which implies that the possible actual path of light propagation between P and Q is the path where the optical path is of the extreme value. The optical path is defined as the geometric path on which the ray travels multiplied by the refractive index of the space where the geometric path is located. As shown in FIG. 1, provided that the ray passes through different media from P to Q through N and M, ΔL_i and n_i are respectively the distance and refractive index of the ray in the i-th medium, then the optical path can be expressed as lPNMQ=$\sum_i n_i * \Delta L_i$. If the refractive index of the medium changes continuously, then the optical path is lPNMQ=$\int_Q^P ndl$. Fermat Principle can be expressed in mathematical language as a variation of the optical path equal to 0, that is, δlPNMQ=0. Besides, the value of the optical path is considered to be the minimum [2].

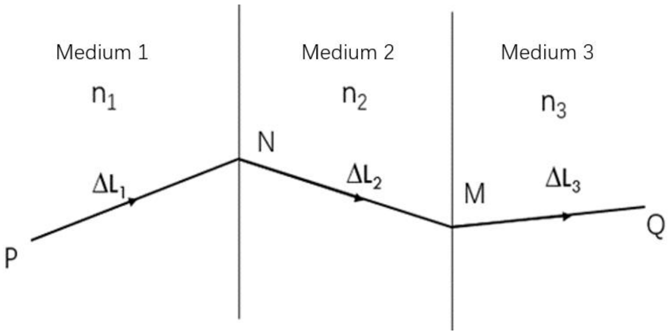

Figure 1. Optical path can be obtained by multiplying the distance of light propagation in different media by the refractive index of the space and then adding the results

For its conciseness and profoundness, Fermat Principle plays the most fundamental and important role in geometry optics [3-7]. However, at present, people deduce geometric optics laws mostly based on the three laws of geometric optics, namely, the optical rectilinear propagation law, reflection law and refraction law, which actually weakens the application of Fermat Principle in practical problems. Thus, this paper elaborates the three laws of geometric optics starting from Fermat Principle in the first place, and then employs a simple geometric relation to assist the principle in deducing the dioptric and catoptric imaging laws of rays on a sphere.

2. Illustration of the three laws of optics from Fermat Principle

2.1 The ray travels in a straight line in a homogeneous medium

In a homogeneous medium, Fermat Principle shows that the optical path between two points is shortest by a line segment. Therefore, the ray travels in a straight line in a homogeneous medium.

2.2 The reflection law of rays

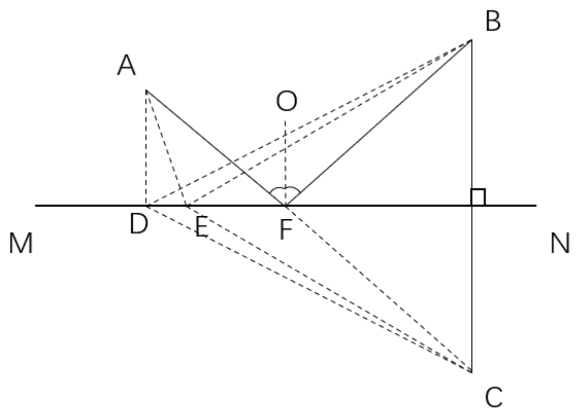

Figure 2. The shortest path of ray reflection is the path when the refraction angle is equal to the incidence angle

Consider a ray emitting from point A to the mirror MN and reflecting to point B. Then according to Fermat Principle, the ray travels along the shortest path while being reflected to point B through the mirror. Therefore, a symmetry point C of B can be found on the other side of the mirror MN, and the path of the ray starting from A to the intersection F of AC and MN and then reflecting back to B from F is the shortest, as what has been shown in FIG. 2. On account of the axiom that the sum of any two sides of the triangle is greater than the length of the third side, it can be figured out that any optical path reflected to B through other points of the mirror is greater than the corresponding optical path reflected through point F. From the geometry of triangles, we can obtain that $\angle AFM=\angle CFN$ while $\angle CFN=\angle BFN$. Thus it is clear that $\angle AFM=\angle BFN$ and $\angle AFD=\angle BFD$ further. In view of this, the reflection law of rays can be generalized as: when a ray is incident from one homogeneous medium to the surface of another homogeneous medium and incurs reflection, the incident ray, normal, and reflected ray are in the same plane and the angle of incidence is equal to the angle of reflection.

2.3 The refraction law of rays

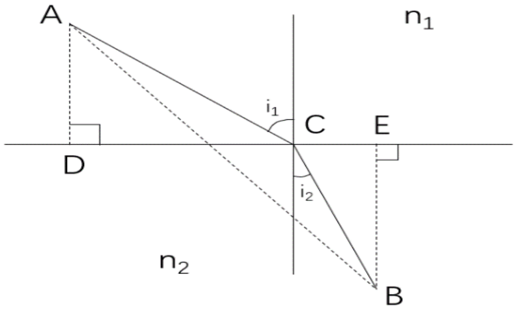

Figure 3. The ray refraction path is the shortest optical path between two points

Consider a ray is incident from point A in medium 1 to point B in medium 2, passing through interface point C. Then cross point A to make a vertical line to the interface point D, and cross point B to make a vertical line interface to point E. Next, make AD=h_1, BE=h_2, DE=p, and DC=x, then it is easy to obtain the optical path, that is $(\mathrm{ACB}) = \mathrm{n}_1(\mathrm{AC}) + \mathrm{n}_2(\mathrm{CB}) = n_1\sqrt{(h_1)^2 + x^2} + n_2\sqrt{(h_2)^2 + (p-x)^2}$, wherein n1 and n2 are the refractive indexes of the upper and lower media respectively. To derive x with the formula, $\frac{d}{dx}(ACB) = \frac{n_1 x}{\sqrt{(h_1)^2+x^2}} - \frac{n_2(p-x)}{\sqrt{(h_2)^2+(p-x)^2}}$ is obtained. Then judging from Fermat Principle, it can be known that the mathematical form of the condition of the minimum optical path value is $\frac{d}{dx}(ACB) = 0$. On this basis, it can be found that $n_1 sini_1 = n_2 sini_2$. Therefore, the refraction law of rays can be expressed as: when a ray is incident from one homogeneous medium to another homogeneous medium and refraction occurs, the incident ray, normal and refracted ray are in the same plane, and the angle of incidence i1, angle of refraction i2 as well as the refractive indexes n1, n2 of the two media meet the following relationship:

$$\frac{sini_1}{sini_2} = n_{21}$$

Wherein, $n_{21}$is the relative refractive index of medium 2 to medium 1, and $n_{21} = \frac{n_2}{n_1}$. Besides, the relation of the refractive index in any medium and the speed of the ray can be expressed as $\frac{c}{v} = n$, wherein, v is the speed of light propagation in the medium and n stands for the absolute refractive index of the medium. Then, define the vacuum absolute refractive index as 1, thus the optical path can be illustrated as the distance of light propagation in the vacuum within the same time as the propagation in the medium.

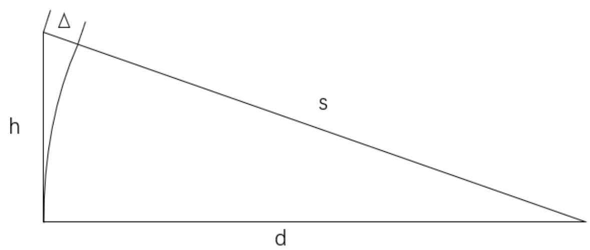

Figure 4. h is much smaller than the difference $\Delta \approx \frac{\mathrm{h}^2}{2s}$ between the two long sides of the right triangle

3. The spherical dioptric and catoptric imaging laws under the paraxial condition of Fermat Principle and a geometric relation

First, note a geometric relationship[8]. The details are as follows: If there is a right triangle with a very small height (h) whose other right-angle side (d) is very long, then how to obtain how much the hypotenuse s is longer than the right-angle side d?

Set the difference $\Delta = s - d$as shown in Figure 4, then we can see $s^2 - d^2 = \mathrm{h}^2$, that is $s - d)\ (s + d) = \mathrm{h}^2$. Since $s - d = \Delta$ and $(s + d) \approx 2s$, so $\Delta \approx \frac{\mathrm{h}^2}{2s}$. This will be the only geometric relation to be applied to the imaging formula using Fermat Principle.

3.1 Convex spherical dioptric imaging

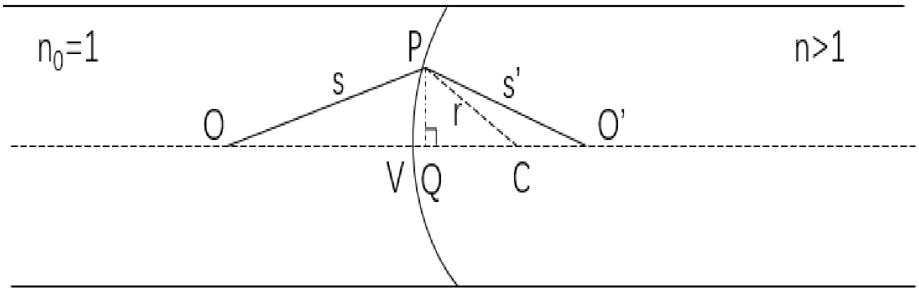

Figure 5. Ray dioptric imaging of the convex sphere

As shown in FIG. 5, a ray is incident from a medium with the left refractive index equal to 1 into the right convex sphere with the refractive index of n, refraction occurs and imaging forms. The imaging condition is that the optical path along the OPO' path is equal to the optical path along the OVO'. Wherein, O is the object point, V refers to the convex spherical vertex, C point is the center of the convex sphere, r is the radius, PQ is the vertical line of the main optical axis, and Q is the vertical point. The the imaging formula can be derived as follows:

Let the optical path be PO=s, PO'=s', and the vertical line be PQ=h. Then, with the Fermat Principle, we can obtain:

$$\begin{aligned} OP - nPO' &= OV - nVO' \\ &= OQ + QV - n(VQ + QO') \end{aligned} \tag{1}$$

By shift, we gain:

$$OP - OQ = (n-1)VQ + nQO' - nPQ' \tag{2}$$

Bring in s, h and r, then we acquire:

$$\frac{h^2}{2s} = (\mathrm{n}-1)\frac{h^2}{2r} + \mathrm{n}(\mathrm{QO'} - \mathrm{PO'}) = (\mathrm{n}-1)\frac{h^2}{2r} - \frac{nh^2}{2s'} \tag{3}$$

By shift, we gain:

$$\frac{h^2}{2s} + \frac{nh^2}{2s'} = (\mathrm{n}-1)\frac{h^2}{2r} \tag{4}$$

After simplifying, we see:

$$\frac{1}{s} + \frac{n}{s'} = \frac{(n-1)}{r} \tag{5}$$

At this moment, when discussing that if the ray is incident from infinity, that is, when s goes to infinity (paraxial ray), the formula is rewritten as:

$$\frac{n}{s'} = \frac{(n-1)}{r} \tag{6}$$

At this time, s’ is called the image focal length, which is recorded as f', then we obtain:

$$\mathrm{f'} = \frac{nr}{n-1} \tag{7}$$

Similarly, when s goes to infinity, the formula is rewritten as:

$$\frac{1}{s} = \frac{n-1}{r} \tag{8}$$

At this time, s’ is called the object focal length and recorded as f, then we see:

$$\mathrm{f} = \frac{r}{n-1} \tag{9}$$

Combined with the above formulas, we get:

$$\frac{1}{s} + \frac{n}{s'} = \frac{1}{f} = \frac{n}{f'} \tag{10}$$

3.2 Concave spherical dioptric imaging

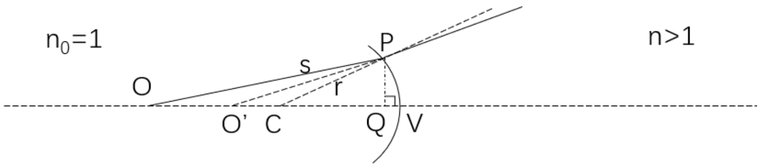

Figure 6. Ray dioptric imaging of the concave sphere

As shown in FIG. 6, a ray is incident from the medium with the left refractive index equal to 1 into the concave sphere with the right refractive index of n (n> 1), and refraction occurs. C point is the center of the concave sphere, r is the radius, PQ is the vertical line of the main optical axis, and Q is the vertical point, Then the imaging formula can be deduced as follows:

Let the optical path be PO=s and the vertical path be PQ=h, then we can know from the Fermat Principle that:

$$\begin{aligned} OP - nPO' &= OV - nVO' \\ &= OQ + QV - n(VQ + QO') \end{aligned} \tag{11}$$

If the virtual optical path is included in the calculation, a negative sign should be added in front of the virtual optical path to calculate the refractive index of the virtual optical path as the refractive index of the medium transmitted by the actual ray.[2]

By shift, we gain:

$$\begin{aligned} OP - OQ &= QV + nPQ' - nVQ - nQO' \\ &= (1-n)QV - n(QO' - PO') \end{aligned} \tag{12}$$

Bring in s, h and r, we get:

$$\frac{h^2}{2r} = (1-n)\frac{h^2}{2r} + n\frac{h^2}{2s'} \tag{13}$$

After simplifying, it is:

$$\frac{1}{s} - \frac{n}{s'} = \frac{1-n}{r} \tag{14}$$

Similarly, we discuss that when the ray is incident from infinity, namely, when s goes to infinity, the formula is rewritten as:

$$\frac{n}{s'} = \frac{n-1}{r} \tag{15}$$

At this point, s 'is like the image focal length and marked as f', then we acquire:

$$\mathrm{f}' = \frac{nr}{n-1} \tag{16}$$

To sum up, it is:

$$\frac{1}{s} - \frac{n}{s'} = -\frac{n}{f'} \tag{17}$$

3.3 Convex mirror catoptric imaging of paraxial rays

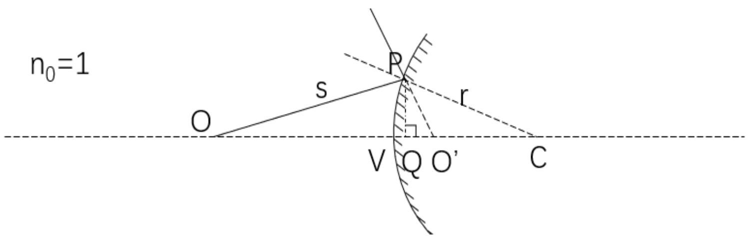

Figure 7. A convex mirror reflection

As shown in Figure 7, a ray is reflected from the left to the convex mirror. Point C is the center of the convex mirror, r is the radius, PQ is the vertical line of the main optical axis, and Q is the vertical point. Then the imaging formula can be deduced as follows:

Let the optical path be PO=s, PO'=s', and the vertical line be PQ=h, then we can know from the Fermat Principle that:

$$\begin{aligned} OP - PO' &= OV - VO' \\ &= OQ - QV - VO' \end{aligned} \tag{18}$$

By shift, we see:

$$\begin{aligned} OP - OQ &= -VO' + PO' - QV \\ &= -2VQ - QO' + PO' \end{aligned} \tag{19}$$

Bring in s, s ', r and h, we get:

$$\frac{h^2}{2s} = -2\frac{h^2}{2r} + \frac{h^2}{2s'} \tag{20}$$

After simplifying, it is:

$$\frac{1}{s} - \frac{1}{s'} = -\frac{2}{r} \tag{21}$$

When s tends to infinity, the formula is rewritten as

$$\frac{1}{s'} = \frac{2}{r} \tag{22}$$

That is:

$$s' = \frac{r}{2} \tag{23}$$

At this point, s 'is the image focal length and marked as f', then:

$$f' = \frac{r}{2} \tag{24}$$

Similarly, when s' goes to infinity, s is the image focal length and marked as f, then:

$$f = -\frac{r}{2} \tag{25}$$

In summary, it is:

$$\frac{1}{s} - \frac{1}{s'} = \frac{1}{f} = -\frac{1}{f'} \tag{26}$$

3.4 Concave mirror catoptric imaging of paraxial rays

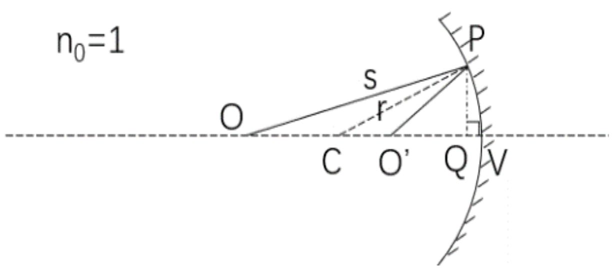

Figure 8. The ray reflection path of a concave mirror

As shown in FIG. 8, a ray is incident from the left to the concave mirror and incurs reflection. Point C is the center of the concave mirror, r is the radius, PQ is the vertical line of the main optical axis, and Q is the vertical point. Then the imaging formula can be deduced as follows:

Let the optical path be PO=s, PO '=s', and the vertical line be PQ=h, then we can know from the Fermat Principle that:

$$OP + PO' = OV + VO' = OQ + QV + VQ + QO'$$

Bring in s, s ', r and h, we get:

$$\frac{h^2}{2s} = 2\frac{h^2}{2r} - \frac{h^2}{2s'} \tag{27}$$

After simplification, it is:

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{r} \tag{28}$$

When s tends to infinity, the formula is rewritten as:

$$\frac{1}{s'}=\frac{2}{r} \tag{29}$$

That is:

$$s'=\frac{r}{2} \tag{30}$$

At this point, s 'is the image focal length and marked as f', then:

$$f'=\frac{r}{2} \tag{31}$$

In summary

$$\frac{1}{s}+\frac{1}{s'}=\frac{1}{f'} \tag{32}$$

4. The unified imaging formulas can be obtained with the help of character rules

For the formulas involved, a fixed character rule is used: let the ray propagates from left to right:

(1) If point O is on the left side of the surface, the object distance s is positive and otherwise, it is negative (the same is true of the object focal length f);

(2) If the point O 'is on the right side of the surface, the image distance s' is positive, and otherwise it is negative (the same is true of the image focal length f ');

(3) If the spherical center C is on the right of the surface, the radius r is positive and otherwise it is negative.

Then we can obtain a unified catoptric imaging formula $\frac{1}{s}+\frac{1}{s'}=\frac{2}{r}$ and a unified dioptric imaging formula $\frac{1}{s}+\frac{n}{s'}=\frac{n-1}{r}$.

5. Conclusion

This paper starts from the Fermat Principle to elaborate the three laws of geometric optics, and then briefly derives the formulas for spherical dioptric and catoptric imaging of paraxial rays through a simple geometric relation. Furthermore, under the regulation of the character rules, it is found that the paraxial spherical refraction and reflection of rays can be summarized by a formula respectively. The exposition and derivation process of this paper fully demonstrates the unique advantages and wide applicability of Fermat Principle in solving geometric optics problems. Reasonable use of the principle can not only make it more concise and direct to deduce the laws of geometric optics, but also provides the optical imaging formula a unified mathematical expression from the same law. In addition, it generates more natural and profound physical meanings, which helps to stimulate the interest of learners. Since the derivation process does not involve calculus, this paper is more conducive to deepening the understanding of geometric optics among high school students and junior undergraduate and junior college students with weak foundation of calculus, thus encouraging them to think more deeply.

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