Study on Timing Characteristic of Saxon Bowl and Prediction of Sinking Time

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Abstract. There have been many water clocks in history and the Saxons used bowl with a hole in its base is one of them. When the bowl is put into water, it will sink slowly. This paper mainly studies series factors that affect the time bowl sinks, water filling rate and how the bowl measures time by establishing a theoretical model, calculating an approximate expression of water filling rate and using ordinary differential equation group for numerical simulation. The effects of aperture of the orifice, weight of the bowl, external height of the bowl on the sinking time were investigated. The innovation of this paper is to find the depth of water inside the bowl can be used as a timekeeping criterion.

Keywords: Saxon bowl; approximate expression; numerical simulation.

1. Introduction

Before devices that can measure a long period of time like the mechanical clock or the electronic clock invented, people used a lot of simple and interesting things to measure time. A timing bowl used by Saxons is one of the water clocks which has an orifice in the middle of the bottom of the bowl. The bowl will sink when placed onto the surface of water smoothly because water will get through the orifice to fill the bowl. Bowls investigated here are cylindrical bowls which can measure not only fixed time characteristics but also continuous timing. The sinking time of the bowl first depends on the rate of water injection. The larger aperture, the greater mass, the lower height, the smaller bottom area, the shorter sinking time. The shape of orifice has little influence on the flow rate because the contraction coefficient and the flow rate equation doesn't have great changes, so it has little influence on the sinking time [1]. And the linear relationship between the water depth inside the bowl and time is good, so we can consider it as a timekeeping criterion. Calculations of water filling rate, sinking time and conclusions can be shown in the research.

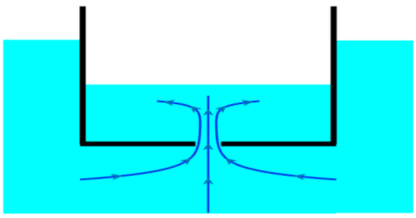

Fig. 1 Saxon timing bowl schematic

Research results show that the negative second power of the aperture is proportional to the sinking time [2]. This result is not consistent with poiseuille flow, so it's not precise to use the pipe flow because the length of the orifice is not long enough; the result of solving the bowl orifice velocity by using Bernoulli equation of submerged outflow at orifice which is the same as the assumption of steady flow accords with the research result. The results obtained by using Bernoulli equation of submerged outflow at orifice have a clear directive significance for studying the timing basis and making Saxon bowls. The variation of water depth in the bowl with time can be predicted accurately after correcting the flow rate equation.

The general idea is to obtain the dynamic equations of the bowl and the water in the bowl, solve the equations to observe the characteristics of the solutions and analyze the various effects of the parameters. The rate of momentum change is easy to write when obtaining the flow rate. Forces on the bowl are mainly gravity, buoyancy and resistance. Gravity and buoyancy mainly make the bowl vibrate up and down can be found when correcting the expressions of forces. So observing the draft depth of the bowl to know the time sometimes is not the best choice. Resistance that proportional to velocity or powers of velocity slows down the vibration and this term has little influence on the variation of water depth in the bowl with time if in normal range.

In this paper, the factors that affect the sinking time of the bowl are discussed, mainly including the influence of aperture of the orifice, inner and outer diameter of the bowl, weight of the bowl, external height on the sinking time of the bowl.

2. Theoretical analysis

Assuming that Saxon bowl is uniformly dense and rigid and doesn't tilt in the water. Saxon bowl needs an easily observable feature that is proportional to time to satisfy the timing purpose. The height of internal and external liquid level is the most easily observed physical quantity. Considering the symmetry of rotation and timing function of Saxon bowl, the symmetry under water surface needs to meet the parallel between the bottom of the bowl and the static water surface, that is, it does not tilt.

2.1 Tilt

Due to perturbation and placement errors in the actual experiment, the motion of the bowl in the water may include tilt or the axis is not at an angle of $90°$ from the horizontal plane. Since it is difficult to judge the depth of water in the bowl after tilting and slow down the rate of water injection, it is necessary to ensure that the bowl will not keep tilting all the time during one rise or fall when making the bowl.

To some extent, the center of mass of the bowl and the water in the bowl is lower than the combined center of the buoyancy and viscous forces when at equilibrium position.

2.2 Rate of momentum change

Considering bowl and water inside it as a whole can simplify the analysis process. To get the momentum equation of the bowl and water inside the bowl, the rate of momentum changes and forces were analyzed separately. Rate of momentum change can be simply obtained from Newton's second law of objects with varying mass.

$$\frac{dp}{dt} = (M + m)\ddot{H} + \dot{m}(\dot{H} - V) \tag{1}$$

$$m = \pi R^2 h \rho \tag{2}$$

Where M is the mass of the bowl, m is the mass of the water inside the bowl, H is the draft depth of the bowl, h is the depth of the water inside the bowl, R is the inner radius of the bowl and V is the velocity of water at the orifice. The term $\dot{m}$ can be given by the flow of water through the orifice.

2.3 Orifice flow velocity and flow rate

The bowl moves slowly most of the time, and the water is approximately considered steady. To simplify the analysis progress, assume that a streamline enters the bowl from outside the bowl, and the flow rate at the height of the orifice outside the bowl is approximately 0. The velocity at the orifice given by Bernoulli equation of submerged outflow at orifice [1] [3]

$$\rho g H = \rho\left[gh - \frac{(\dot{H} - \dot{h})^2}{2}\right] + (\zeta_1 + \zeta_2)\frac{\rho V^2}{2} \tag{3}$$

We can get

$$V = \sqrt{\frac{2g(H-h)+(\dot{H}-\dot{h})^2}{\zeta_1+\zeta_2}} \tag{4}$$

Where ρ is the density of water, r is the aperture of the orifice, g is the acceleration of gravity, ζ1and ζ2 are respectively kinetic energy correction factor and local resistance coefficient of sudden expansion of liquid after contraction section. The resistance coefficient of the thin-walled orifice is usually taken here cause the aperture is generally large [1]. $(\dot{H}-\dot{h})^2/2$ is a pressure correction term modified by Bernoulli equation?

Flow rate equation can be written as

$$\dot{h} = \frac{r^2}{R^2}\varepsilon(V+\dot{H}) \tag{5}$$

Where ε is the contraction coefficient at the orifice, the average shrinkage coefficient of sinusoidal pulsating flow is slightly smaller than that of quasi-steady flow method, but the fluctuation time is short and the effect is smaller [1] [4].

2.4 Dynamic equation

Forces on the bowl simplified are gravity, buoyancy, viscous resistance and flow resistance. Momentum equation of the bowl and water inside the bowl can be written as

$$(M+m)\ddot{H} + \dot{m}(\dot{H}-V) = (m+M)g - \pi R_1{}^2\rho\left(gH - \dot{H}^2/2\right) - k\dot{H}\pi R_1{}^2 \tag{6}$$

Where R1 is the outer radius of the bowl. $\dot{H}^2/2$ is a pressure correction term. Sinking time of the bowl is sensitive to the thickness of the wall. To satisfy the bowl sinks at a constant rate, minimize errors caused by bowl thickness and ensure the bowl doesn't tilt easily. Reducing the thickness of the bowl is important. Here we write the flow resistance and viscous resistance together as $k\dot{H}\pi$R12 which is proportional to $\dot{H}$, and this term reduces the vibration the most though has little impact on sinking time.

2.5 Solution

Two differential equations relate the speed, acceleration of the bowl and the depth of the water inside the bowl.

$$\ddot{H} = g - \frac{\pi R_1^2\left[\rho\left(Hg - 0.5\dot{H}^2\right) + k\dot{H}\right] + \dot{m}(\dot{H}-V)}{M+m} \tag{7}$$

$$\dot{m} = \rho\pi R^2\dot{h} \tag{8}$$

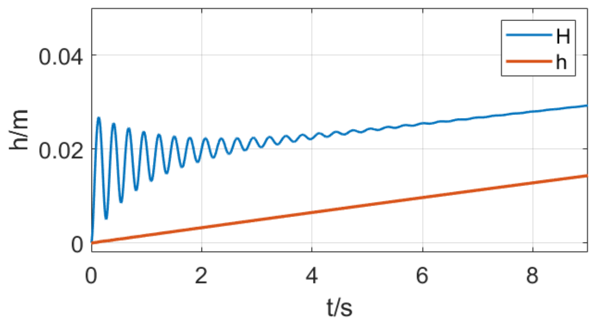

Fig. 2 Diagram of draft depth and depth of the water inside the bowl changing with time.

There is no analytical solution to this differential equation set. We use MATLAB software for numerical solution, using ODE45 function, which is the fourth-order - fifth-order Runge-Kutta algorithm. The initial condition is initial height inside the bowl and the initial speed is 0 and the depth of the draft is also 0 mathematically to approach the real condition. Plot one of the numerical solutions of the device over time.

The curve above reflects the draft depth and the curve below is the depth of the water inside the bowl. Numerical solution can predict H and h accurately, so we can find the points that H is greater than the height of the bowl or h is greater than a certain height to get the sinking time. By observing the numerical solution obtained from the parameters of the input experimental materials, it can be found that h is approximately a straight line, and H also becomes a straight line when t is large. The theory can predict $\dot{h}$ most accurately, so we can use constant $\dot{h}$ to predict the sinking time of bowls. We can assume that $\dot{h}$, $\dot{H}$ and $H-h$ to be constant separately and put in the differential equation. Constants are

$$H-h=a;\dot{H}=\dot{h}=b \tag{9}$$

Then b can be solved as

$$b=\frac{\sqrt{k^2S_1^2+4\left[Mg-(S_1-S)\rho gH_0\right]\left(\rho S_k+{S\rho a_3\xi^2}/{2a_1}\right)}-kS_1}{2\rho\left(S_k+{Sa_3\xi^2}/{2a_1}\right)}\approx\sqrt{\frac{2Mga_1}{S_1\rho a_3\xi^2}} \tag{10}$$

Defined parameters are

$$\varepsilon_s=\varepsilon\frac{r^2}{R^2};\ \xi=\sqrt{\zeta_1+\zeta_2};$$

$$a_1=\frac{1}{{\varepsilon_s}^2}-\frac{1}{\xi^2};a_2=\frac{1}{\varepsilon_s}+\frac{1}{\xi^2};a_3=(a_1-a_2)^2-\frac{a_2^2}{a_1}-1+\frac{1}{\xi^2};\ S_k=\left(2-\frac{1}{\varepsilon_s}\right)S-\frac{S_1}{2}$$

Where S and S_1 are respectively the area of the bottom inside and outside of the bowl, H_0 is the height of the bowl. KS_1 on the molecule is much smaller than the term containing mass, the base area corrected buoyancy term is much smaller than the gravity term, the S_k in the denominator is much smaller than the latter term, the S and S_1 is unified and replaced by S_1, then this approximate formula can be obtained. Because b is greater than 0, we drop the negative solutions. This approximation of b is accurate.

Just before the bowl is submerged, there is a certain height difference between water inside and outside of the bowl, and the water depth inside the bowl is defined as the maximum depth inside the bowl, which is called h_m. When h reaches h_m, the bowl stops timing and sinking time T_{total} can be calculated as

$$T_{total}=\frac{h_m}{b} \tag{11}$$

We can consider water depth inside the bowl to be h_m at the point just before submerging if the bowl moves down directly and slowly all the time.

Now we have the relationship between sinking time T_{total} and parameters of the bowl.

3. Experiment

To enable the experiment to be repeated in parallel, experiments were done by placing the bowl smoothly into the water. When placing the bowl, the bottom of the bowl should be slightly in contact with the water surface evenly. Observe the hole of the bowl to ensure that water does not enter the hole.

The single variables that my apparatus can satisfy are aperture and bowl mass and we can observe the numerical solution of change rate of water depth inside the bowl varies with each variable. Data from other apparatus can be used to verify the theory.

3.1 The linear relationship between h and t

Select two bowls to draw the scatter plot of experimental data of h changing with time and $h = bt$ line to compare.

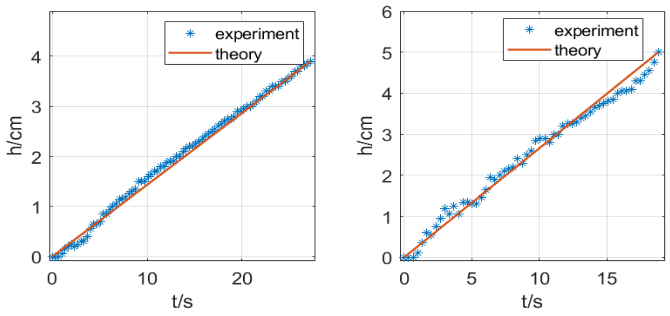

Fig. 3 linear relationship between h and t

The determination coefficients of the two groups were 0.9957 and 0.9920, which shows the linear relationship between time and h is good.

3.2 Effect of device parameters

To study the effect of aperture and mass, we can observe the change rate expression of h, make the relation graph of aperture radius square with $\dot{h}$ and mass to -0.5 power with sinking time. $\dot{h}$here is average rate of water injection. Keep other input parameters consistent with the experimental equipment and make the experimental data and numerical solution graph.

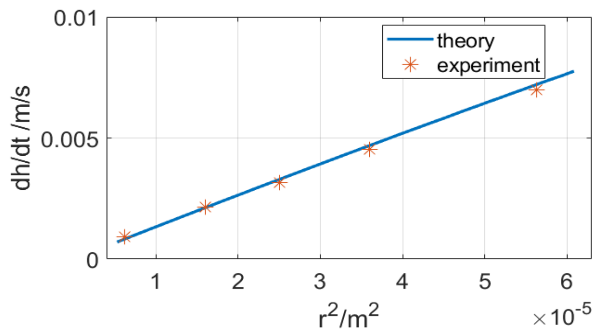

Fig. 4 Effect of aperture on $\dot{h}$

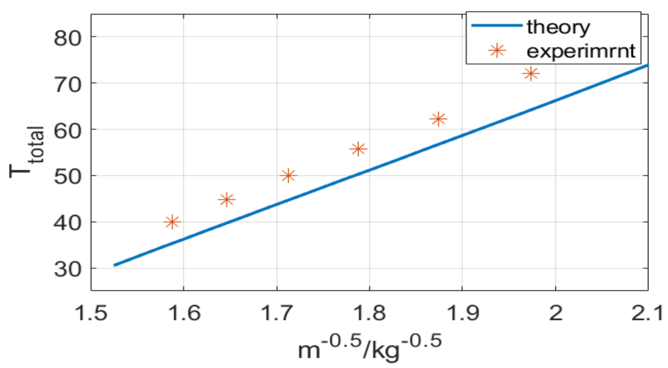

Fig. 5 Effect of mass on T_{total}

The results show that there is a good linear relationship between water filling rate and aperture square, mass to -0.5 power and sinking time, which is consistent with the expression.

4. Conclusion

A theoretical model is established by connecting dynamic equation and flow rate equation, and then expression of rate water fills and numerical solution of the height of internal and external liquid level with time is obtained. Rate water fills and sinking time calculated by numerical solution are compared with experiment, conclusions are obtained:

1. The depth of water inside the bowl has a better linear relationship with time than the draft depth of the bowl and the depth of water inside the bowl can be used as a timekeeping criterion is the innovation of this paper.

2. The rate of water injection is approximately proportional to the negative second power of the aperture, 0.5 power of the mass.

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