Transition from complex numbers to complex function and series.

Xiling Wang*

Chengdu Experimental Foreign Language School

*Corresponding author: 100758@yzpc.edu.cn

Abstract. Complex analysis, traditionally known as the theory of functions of a complex variable, is a branch of mathematical analysis that investigates functions of complex numbers. If n is odd, there is only one order n root, and if n is even, there are only two order n roots. Things change in complex numbers, though. Keep in mind that complex numbers have closed algebra. As a result, there exist n roots of order n always.Dating back to the 16th century when Italian mathematicians Girolamo Cardano and Raphael Bombelli first observed complex numbers when they were trying to solve a certain algebra and developed by Cauchy and Riemann in 19th century, complex analysis nowadays has played a more and more importance role on mathematics, physics and engineering, especially in the fields of algebraic geometry, fluid dynamics, quantum mechanics and so on. In this paper, the birth and discovery of complex number will be discussed. Starting from discovery of complex number, the properties of complex function and power series are elaborated which are foundation for mordern mathematical conjectures.

Keywords: Complex numbers, complex function, power series.

1. Introduction

Is the existence of complex numbers ensured yet? Is this a hard issue? It is infeasible to derive the square roots of negative numbers from definition and state that there does exist these kind of numbers. With the exception of representation, the problem was under-solved til approximately 30 decades ago. No matter if complex numbers have any real meaning in the real world, they are far more than values that breed from considerable mathematical disciplines [1-15].

For years and years, ancient mathematics developed rapidly with little concerns about the restrictions of just real positive values. The original exploration of imaginary numbers was just not an Archimedean “Eureka” moment back in the middle of the 16s century [1-15]. Mathematicians lingered over some similar conceptions about complex numbers for ages. Ancient mathematicians had no knowledge about negative numbers, do not have to say imaginary numbers. However, experts who specialized in solving cubic equations recognized something crucial - “impossible values”. The conception was so strange that they believe that they had found something distinct. Fundamentally, there always existed a “linguistic barrier”while mathematicians were working. As a result, conflict became ubiquitous in that era. As there is no such a “ruler” to measure their thinking, mathematicians was constructing conceptions from the very underpinning. Consequently, little progress was made in the complex area for decades. However, the postponed consistency that was reached - the definite existence of a complex system - boosted mathematical researches concerning roots of functions, the Cartesian coordinate system and so on[1-15].

Over 4 centuries, the complex system was studied quite intensely since it became more broadly accepted as mathematical fact, not only for the sake of the early representation but by choice[1-15]. As mathematicians started to query mathematics in advance. The surface of many abstract mathematical proofs led to confidence of mathematicians to continue to study the complex system. Actually, the complex system is studies as a single subject nowadays, namely “complex analysis”. Thereby, the fact that the evolution of the system that was facilitated by the extension of rigorous proof provided brand new and widening perspectives that can help to approach a lot of branches of mathematics. The original tracks of imaginary numbers was discovered in Italy, hidden inside a cubic equation. Date back to the early 1500s, the line for differentiating university mathematics and common mathematics shrunk, in the mean time there was a fast development in algebra[1-15]. By 1530s, a genius in mathematics called Nicole Tartaglia arrived in this area. Tartaglia was quite interesting and sometimes being called “the stammerer” for a speaking disability due to his hometown-massacre in his childhood [1-15]. Anyways, the accomplishments of Tartaglia was overlooked as well as his ideas in mathematics.

One of his most important achievements was the introduction of a particular method for a certain type of cubic equations. It is then needed to examine Tartaglia’s skill. Although it had a firmly expository style, it was a breakthrough in that era in the algebraic scene. The method started to apply abstract logic rather than numerical samples, mitigating the question, “ Is this a general method?” Tartaglia adopted a method of Diophantine technique, leading to representation of multiple equations for his scenario. His solution was written in a demonstrative form, which is also called Tartaglia’s Poem. Start with examining the decreasing cubic equation

$x^3 + cx = d$. To proceed, Tartaglia gave a definition to the two constants so that the difference of which is d and their product is the same as one-third of C cubic. Let these numbers be u and v. Therefore, u - v = d, uv= (c/3)^3. The task is still there today: derive u and v with their product and difference provided. This contains the process of solving quadratics that mathematicians had worked on the method to solve. Tartaglia pursues the value of the sum with the aid of the existing method, in order to apply this along with u - v to solve the values of each variable in a source of systematic equations. Tartaglia yields u + v through squaring the difference, plus their product times four, and calculating the square root of the value. He then answers that the solution x = u^1/3 - v^1/3. Plugging in some constants to do an application. To start with the form of equation in $x^3 + 3x = 4$

To derive the solution, it is essential to find the value of u and v where their difference and product is 4 and 1 respectively. Therefor, with Tartaglia’s method, it is then handy to find u = v from above

2. Main works

2.1. Complex numbers and the complex plane

Define the set of complex numbers by $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$. Besides, $i^2 = -1$, which means $i = \sqrt{-1}$. For a complex number $z = x + iy$, it is further defined that $\mathrm{Re}(z) = x$, so called the real part $\mathrm{Im}(z) = y$, the imaginary part, $|z| = \sqrt{x^2 + y^2}$, the mod of z. It is said that z is real if $\mathrm{Im}(z) = 0$, and z is said to be imaginary if $\mathrm{Re}(z) = 0$.

Algebra in the $\mathbb{C}$ plane:Let $z_1 = x_1 + iy_1$,$z_2 = x_2 + iy_2$

i. Addition $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

ii. Multiplication $z_1 z_1 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_1 - y_1 y_1) + i(x_1 y_2 + x_2 y_1)$

iii. Conjugation: $\overline{x + \imath y} = x - iy$

The operation of adding and multiplying is exactly the same as algebra in the $\mathbb{R}$ plane

Just to separate the terms with I and those not and to calculate respectively with also applying $i^2 = -1$. Whatsoever, $\mathbb{C}$ is then made in to a field where adding and multiplying are relative, cumulative, and it admits identities like 0 and 1 respectively as well as the inverse and reciprocal $-z$ and $\frac{1}{z}$ where $z \neq 0$, and satisfy some distribution. It is also verified that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$; $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$, $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$, $|z| = \sqrt{z\bar{z}}$.

For all numbers on the complex plane, identity * shows $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ $(z \neq 0)$. Another way to come to this is to take the conjugate of the denominator. $\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{\overline{x+\imath y}}{|x+iy|^2}$ With which it can be derived that $|z_1 z_2|^2 = (z_1 z_2)(\overline{\overline{z_1 z_2}}) = z_1 z_2 \overline{z_1}\,\overline{z_2} = |z_1|^2 |z_2|^2$. Therefore,$|z_1 z_2| = |z_1||z_2|$. In addition, when $\bar{z} = z$, z on the complex plane is a real number, correspondingly, z is imaginary when $\bar{z} = -z$.

To put $\mathbb{C}$ on paper. $\mathbb{C}$ shall be presented as a right angle coordinate where z=x+iy is represented by a pair of coordinates (x, y). Thereby, the x-axis is named to be the real axis whilst the y-axis is the imaginary axis, instead of the Cartesian coordinate, this is called the complex plane.

By calling back the method of yielding distance between two points, it is obvious that the mod of z is the distance between z and the origin. To be more general, the mod of $z_1 - z_2$ is the distance from z_1 to z_2. It shall be noticed that addition can be operated through vector space, and then it can be seen in another way. To do the same thing to multiplication, firstly every point is suggested to be written in the way of polar coordinates. $\left|\frac{z}{|z|}\right| = |z|\left|\frac{1}{z}\right| = |z|\frac{1}{|z|} = 1$. Then there exist $\theta \in \mathbb{R}$ were $\frac{z}{|z|} = \cos\theta + i\sin\theta$. Let $e^{i\theta} = \cos\theta + i\sin\theta$, therefore, $z = |z|e^{i\theta}$

Define the polar form of $z \in \mathbb{C}$ $: z = |z|e^{i\theta}$ where θ as well as $\theta + 2k\pi \in \mathbb{R}$ is called the argument of z, the notion of which is simplified to be $\arg z$, θ is the principle value, $\operatorname{Arg} z$.
By applying trigonometric identities, it can be checked that $z_1z_1 = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Then for $z_1, z_2 \in \mathbb{C}$, $z_1z_2 = |z_1|e^{i\theta_1}|z_2|e^{i\theta_2} = |z_1z_1|e^{i(\theta_1+\theta_2)}$. Therefore the process from z_1 to z_1z_1 involves expanding or narrowing z_1 by z_2 and rotation anticlockwise by θ_2. Particularly, the process from z to iz represents an anticlockwise rotation of $\arg i = \frac{\pi}{2}$. Lastly, complex conjugation represents a reflection over the real axis. To convince one the existence of I and the complex plane, applying a 2×2 matrix is a suggested method, $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$ can be considered. Inequalities concerning comples numbers is discussed as follows.

As the mod of z represents the the distance between the point and the origin, the triangle inequality can be derived $|z_1 + z_2| \le |z_1| + |z_2|$.

Therefore, it is implied the other triangle inequality $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$

Then $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$. Thereby $|z_1| - |z_2| \le |z_1 - z_2|$

Applying the same manipulation $|z_2| = |z_2 - z_1 + z_1| \le |z_2 - z_1| + |z_1|$

Then $-(|z_1| - |z_2|) = |z_2| - |z_1| \le |z_2 - z_1| = |z_1 - z_2|$, $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$

Finally $|\mathrm{Re}(z)| \le |z|$ & $|\mathrm{Im}(z)| \le |z|$

Convergence of series: By definition, a sequence $(z_n)_{n\in\mathbb{R}} \subset \mathbb{C}$ converges to $\omega \in \mathbb{C}$ if $\lim_{n\to\infty}|z_n - \omega| = 0$, noted as $\lim_{n\to\infty} z_n = \omega$, where ω is called the limit of the sequence. More specifically, $(z_n)_{n\in\mathbb{R}}$ is said to converge to ω if for any $\varepsilon > 0$ there exsists $N \in \mathbb{N}$ so that for any $n \ge N, |z_n - \omega| < \varepsilon$.

As for $|\mathrm{Re}(z)|, |\mathrm{Im}(z)| \le |z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$, $\lim_{n\to\infty} z_n = \omega$. Therefore, $\lim_{n\to\infty} \mathrm{Re}(z_n) = \omega$ and $\lim_{n\to\infty} \mathrm{Im}(z_n) = \omega$. Then the notion here is consistent with the common one on the real plane.

2.2. Functions on the complex plane

It is always taken into consideration the regularity, differentiability and continuity of a function. Let $\Omega \subset C$, it is defined that function f: $\Omega \subset C$ is continuous at a point $z_0 \epsilon \Omega$ if for every $\varepsilon > 0$ $\exists\, \delta > 0$ that no matter when $z \epsilon \Omega$ gratifies $|z - z_0| < \delta$, then it can be derived that $|f(z) - f(z_0)| < \varepsilon$. The function f is said to be continuous on Ω if it is continuous at every $z_0 \epsilon \Omega$. Equivalently, f is continuous at the point z_0 if all sequences $(z_n)_{n\epsilon N} \subset \Omega$ converging to z_0, the series $(f(z_n))_{n\epsilon N}$ converges to $f(z_0)$.

A function f is defined to be able to reach its maximum/minimum at point $z_0 \epsilon \Omega$ if $|f(z)| \le |f(z_0)|$/$|f(z)| \ge |f(z_0)|$ for all $z \epsilon \Omega$. A very useful feature of analyzing of a compact set is the following theorem, the proof of whom is deferred from the real analysis course. A continuous function on a compact set $\Omega \subset C$ is bounded and reaches a maximum and a minimum. Let $\Omega \subset C$ be open and impose a function f: $\Omega \rightarrow C$. The function f is said to be holomorphic at $z_0 \epsilon \Omega$.if there exists the limit $\lim\limits_{z\longrightarrow 0}\frac{f(z0+z)-f(z0)}{z}$ .To be specific, if there exist a complex w that belongs to C, therefore for all any real number bigger than zero there exists a delta that bigger than zero and that if h gratifies h is bounded by delfta then is can be obtained that z+h that indside region as well as the limit $\lim\limits_{z\longrightarrow 0}\frac{f(z0+z)-f(z0)}{z}$ exist. However, this definition is far more potent, which means it can be seen whether f is holomorphic or not on an open set then it is said to be infinitely differentiable there. Function f: $\Omega \rightarrow \mathbb{C}$ is holomorphic only if $\exists$ a ϵ $\mathbb{C}$ and $r > 0$ therefore for $|h| > r$, $\mathrm{f}(z_0 + h) = f(z_0) + ah + h\varphi(h)$ in which $\varphi\colon D_r(0)\backslash\{0\} \rightarrow \mathbb{C}$ gratifies $\varphi(h) \rightarrow 0$ as h approaches o. So one can have $a = \mathrm{f}'(z_0)$

Proof:Let $a = \mathrm{f}'(z_0)$ and select an $r > 0$ thereby $D_r(z_0) \subset \Omega$. Then $\varphi(h) := \frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)$ for any $h\epsilon D_r(0)\backslash\{0\}$. So that $\lim\limits_{h\rightarrow 0}\varphi(h) = 0$ of f is holomorphic at the point z_0 by definition. Which means, given $\varepsilon > 0$, introduce a δ where $0 < \delta < r$ that $|\varphi(h)| < \varepsilon$ for $|h| < \delta$. Then $\left|\frac{f(z_0+h)-f(z_o)}{h} - a\right| = |\varphi(h)| < \varepsilon$. The lemma above indicates the implication that being holomorphic at a point may mean by being continuing in the meantime.

If both f and g are holomorphic on Ω, it can be obtained that

$(f + g)' = f' + g' \Longrightarrow f + g$ is holomorphic on Ω

$(fg)' = f'g + fg' \Rightarrow fg$ is holomorphic on Ω

$\{z \in \Omega\colon g(z) = 0\}$ & $\left(\frac{f}{g}\right)' = \frac{f'g-fg\prime}{g^2} \Rightarrow \frac{f}{g}$ is holomorphic on Ω

Whatsoever, if function f has the feature $\Omega \rightarrow u$ and for g one has $u \rightarrow \mathbb{C}$ and they are both holomorphic, consequently $g\circ f$ is holomorphic on Ω as$(g \circ f)'^{(z)} = g'(f(z))f'(z)$. The proofs of the above three statements are the same as those in real analysis, so only the proof for the statement will be provided. Let $z_0 \in \Omega$. Applying the lemma mentioned above, then one has function φ and ϕ where $\mathrm{f}(z_0 + h) = f(z_0) + f'^{(z_0)} + h\varphi(h)$ & $\mathrm{g}(f(z_0 + k)) = g(f(z_0)) + g'^{(f(z_0))}k + k\phi(k)$. For h and k that are small enough and has the feature $\varphi(h) \rightarrow 0$ when $h \rightarrow 0$ & $\phi(k) \rightarrow 0$ when $k \rightarrow 0$. $\mathrm{g} \circ \mathrm{f}(z_0 + h) - \mathrm{g} \circ \mathrm{f}(z_0) - g'^{(f(z_0))}f'^{(z_0)} = g\left(f(z_0) + f'^{(z_0)}h + h\varphi(h)\right) - g(f(z_0)) - g'^{(f(z_0))}f'^{(z_0)} = \mathrm{g}(f(z_0)) + g'^{(f(z_0))}k + k\phi(k) - g(f(z_0)) - g'^{(f(z_0))}f'^{(z_0)}h = \mathrm{g}'^{(f(z_0))(k-f'^{(z_0)}h)} + k\phi(k) = \mathrm{g}'^{(f(z_0))}h\varphi(h) + \left(f'^{(z_0)}h + h\varphi(h)\right)\phi\left(f'^{(z_0)}h + h\varphi(h)\right) = \mathrm{h}\left[g'^{(f(z_0))}\varphi(h) + \left(f'^{(z_0)} + \varphi(h)\phi\left(f'^{(z_0)}h + hf(h)\right)\right)\right]$.

As the value of the square brackets approaches 0 while h tends to 0, the mentions lemma shows $g \circ f$ is holomorphic at z_0 for the provided derivative.

Cauchy-Riemann Equations

Let $\mathrm{f}\colon \Omega \rightarrow \mathbb{C}$, then define functions with real value $\mathrm{u}(x,y) := Re(f(x + iy))$; $\mathrm{v}(x,y) := Im(f(x + iy))$ Therefore, $\mathrm{f}(x + iy) = u(x,y) + iv(x,y)$. Assume that f is holomorphic at the point $z_0 = x_0 + iy_0$. Considering the partial fraction of $\mathrm{f}(x,y) := f(x + iy)$ on $\mathbb{R}$

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{\mathbb{R}\ni h\rightarrow 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \tag{1}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) := \lim_{\mathbb{R}\ni h\rightarrow 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \tag{2}$$

Relate it with $\mathrm{f}'^{(z_0)}$ via using different paths of $\mathbb{C} \ni h \rightarrow 0$. Firstly, one have

$$\mathrm{f}'^{(z_0)} = \lim_{\mathbb{R}\ni h\rightarrow 0}\frac{f(z_0+h,y_0)-f(z_0)}{h} = \lim_{\mathbb{R}\ni h\rightarrow 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \frac{\partial f}{\partial y}(x_0, y_0) \tag{3}$$

Then $i\mathbb{R} \ni ih \rightarrow 0$ it follows that

$$\mathrm{f}'^{(z_0)} = \lim_{\mathbb{R}\ni h\rightarrow 0}\frac{f(z_0+ih,y_0)-f(z_0)}{ih} = \lim_{\mathbb{R}\ni h\rightarrow 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} = \frac{\partial f}{i\partial y}(x_0, y_0) \tag{4}$$

It can further be derived that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \;\&\; \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0) \tag{5}$$

Consequently, as f is defined to be holomorphic on an open set $\Omega \subset \mathbb{C}$, it has been shown the real and imaginary part of it having partial derivatives gratifying the Cauchy-Riemann equations on all of Ω. Whatsoever, the partial derivatives would be continuous when the consequence of holomorphic functions are able to be differentiated .

2.3. Power series

Infinite degree of polynomial denoted as: $r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \ldots + r_n x^n = \sum_{i=0}^{\infty} r_i x^i$ where x ∈ C. For power series $\sum_{i=0}^{\infty} r_i x^i$ convergence for given x ∈ C such that partial sum $S_N(x) = \sum_{i=0}^{N} r_i x^i$ converge as N → ∞.

More explicitly:For any $\gamma > 0$, there exist a number ρ and N(ε) such that:$\left|\sum_{i=0}^{\mathrm{N}(\varepsilon)} r_i x^i - \rho\right| < \gamma$, denoted as: $\sum_{i=0}^{\infty} a_i z^i = \rho$.For power series $\sum_{i=0}^{\infty} r_i x^i$ absolutely convergence for given x ∈ C if $\sum_{i=0}^{\infty} |r_i x^i|$ convergent. For power series $\sum_{i=0}^{\infty} r_i x^i$, there exist the radius of convergence r $= \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}$ such that when |x| < r, it is absolutely convergence and when |x| > r, it is divergence. The region |x| < r as the disc of convergence. f' has the same radius of convergence as f.The sequences of function $\{r_n\}$ converges uniformly when for given ε, there exists N(ε) such that when i ≥ N, $\| f_i - f \| < \varepsilon$. The Cauchy sequence $\{r_n\}$ satisfy the property that: $| r_i - r_j | \to 0$ as i, j → ∞. More explicit: For any $\gamma > 0$, there exist integer $\sigma = \sigma(\gamma)$ such that:$|r_i - r_j| < \gamma$ for any i, j > $\sigma(\gamma)$. The Cauchy sequence $\{r_n\}$ is uniformly convergence. When $\{a_i\}$ be a convergent sequence with all elements are nonnegative real numbers and $\{b_i\}$ be s sequence such that $\|b_i\| \leq a_i$ for all i ≥ 0. Then $\sum b_i$ uniformly and absolutely convergent. Let $\{r_n\}$ be a sequence of complex numbers, and there exist positive number k such that:

$\sum |r_n| k^n$ converges. Then the series $\sum r_n i^n$ is absolutely and uniformly convergent for |i| ≤ k. Let $\{r_n\}$ be positive number sequence and $\lim \frac{r_{n+1}}{r_n} = k \geq 0$, then $\lim a_n^{\frac{1}{n}} = k$.
The function f on set γ is called the analytic at point $x_0 \in \gamma$ if there exists power series $\sum a_n (x - x_0)^n$ with radius of convergent centered at x_0.

3. Summary

The development and discovery of complex numbers will be covered in this essay. The features of complex function and power series are developed starting with the discovery of the complex number, laying the groundwork for more recent mathematical conjectures. Complex numbers were created by resolving equations with degrees greater than one. Prior to the nineteenth century, the majority of mathematicians concentrated on real analysis and its applications in physics. The first person to make significant attempts in complicated analysis was Cauchy. By presenting his integral formula, he computed a number of complex integrals and laid down the basis for defining the operations and features of complex analysis. The geometrical theory of complex numbers was further developed by Gauss. The fundamental structure of complex analysis was nearly complete once Riemann presented the idea of the Riemann surface. It was soon used in physics and other fields. Additionally, the conformal mappings and holomorphic function theories have an impact on quantum field theory and string theory. Another lovely idea derived from complex analysis is modular form. But what distinguishes modular form from other forms is its connection to elliptic curves, which links complex analysis to number theory or arithmetic geometry. In the future, more techniques regarding complex function will be reported.

References

[1] Liu Shenghua, Pan Jifu, Zheng Jiyun, complex change function [M]. Changchun: Jilin Education Press,1988。

[2] Zhong Yuquan, complex function Theory (second edition) [M]. Beijing: Higher Education Press,1988.

[3] L V Alforth, complex analysis [M]. Shanghai: Shanghai Science Press, 1984.

[4] Tan Xiaohong, Wu Shengjian complex change function concise tutorial [M]. Beijing: Peking University Press,2006.

[5] Jerrld E Maislen, Basic complex analysis[M]. Freeman W H and Company, 1973.

[6] A Simple Proof of the Fundamental Cauchy-Goursat Theorem, Eliakim Hastings Moore, American Mathematical Society.

[7] Complex variables and applications / James Ward Brown, Ruel V. Churchill.—9th ed. 1221 Avenue of the Americas, New York, NY 10020.

[8] Matrices, Faculty of Mathematics Centre for Education in Waterloo, Ontario N2L 3G1.

[9] Integration along curves, VED V. DATAR.

[10] The Cauchy–Riemann Equations, Joel Feldman, 2012.

[11] Cauchy's integral formula, Jeremy Orlof.

[12] Taylor's Theorem and Applications, James S. Cook, November 11, 2018, For Math 132 Online.

[13] Taylor & Laurent theorem, Chandan kumar Department of physics S N Sinha College Jehanabad Introduction.

[14] Complex Analysis, Elias M. Stein and Rami Shakarchi, published by Princeton University Press and copyrighted, © 2003.

[15] A Formal Proof of Cauchy's Residue Theorem, Wenda Li and Lawrence C. Paulson Computer Laboratory, University of Cambridge.