

Proof of Cauchy's theorem on Disc

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Abstract. Mathematical analysis known as "complex analysis" examines characteristics of functions, sequences, derivatives of complex numbers, and other objects. When Italian mathematicians Girolamo Cardano and Raphael Bombelli attempted to solve a unique algebra, that is when people first learned about complex numbers. After hundreds of years of growth, complex analysis is now a crucial component of mathematics, physics, and engineering, particularly in the areas of algebraic geometry, fluid dynamics, quantum mechanics, and other related subjects. Complex analysis is a fascinating and esoteric field of study. Mathematicians have created groundbreaking work in this area during the past few centuries. Complex numbers were first defined by mathematicians, who subsequently gradually looked into their algebraic and geometric properties. After hundreds of years, they discovered and studied complex computations. Convergent power series local representations exist for holomorphic functions. This amazing discovery, made by Cauchy between 1830 and 1840, helps to explain the fascinating properties of holomorphic functions. The Cauchy integral theorem is one of the most important concepts in complex analysis. In this paper, a detailed proof of Cauchy's theorem on a local disc is given.

Keywords: complex number, topology, Cauchy theorem.

1. Introduction

In the middle of the 16th century, Cardano demonstrated the format of mathematical proofs but was unable to convey a full mathematical theory. At this time, mathematicians were used to learning mathematics exclusively using what they already understood. It is essentially geometric, similar to Cardano's *Magna*, and the words are not stated precisely, therefore these proofs cannot serve as a strong foundation [1-18]. Cardano referred to almost all of his mathematical justifications as "arguments." *Magna* reveals that mathematics was mostly founded on faith before the emergence of complicated systems. The introduction of imaginary numbers is applicable to cubes and more according to specific arguments. Without a doubt, Raphael Bombelli was a brilliant mathematician. Bombelli is not an expert in math. Instead, he worked as an engineer and architect for the most of his life. He was informed about the Cardano cubic solution. Bombelli, in contrast to Cardano, is unable to explain away his failure. Due to the actual rigor it brought to imaginable numbers, his work *Algebra* marked a turning point in the evolution of complex numbers. According to Bombelli, the amounts obtained by Cardano's approach are not merely indications of partially ineffective methods but also "actual numbers" with the potential to lead to new mathematical discoveries. Bombelli's term is the earliest representation of imaginary numbers as we know them. He determined that $\sqrt{-1}$ is pdm, which translates to "plus or minus." At that time, rather of using symbolic notation, mathematics was still mostly written in literal form. As a result, the way in which mathematical formulas are represented now would be substantially different in Bombelli's language. There is evidence that Bombelli not only came up with the phrase "imaginary numbers," but also understood some of their characteristics and associated operations. He recognized that the identity $i^2 = -1$ In his language, "a negative plus or minus times a negative minus gives a negative," and that "negative" is essentially the negative identity, minus 1. In order to find the complex number's cube root, Bombelli "attacks the irreducible case of a cube." He created a unique approach to solving cubic equations that bravely acknowledged the presence of negative square roots [1-18]. Therefore, Bombelli's method might

herald the birth of complex numbers. Think about Bombelli's developments with the cubed irreducible example. He began by demonstrating how one could not locate these roots using Cardano's formula. He did, however, demonstrate how combining fictitious roots can result in actual numbers. We can see that the real numbers Bombelli illustrates are produced by complex numbers when viewed in a completely different way. Think about the cubic equation provided by $a^3 = 15a + 4$. The answer is obtained by solving this problem using Cardano's method. Bombelli acknowledges that none of the three origins. Bombelli went on to demonstrate in 1560 that a real number might be created by manipulating the fictitious phrase provided by Cardano's method. Bombelli claims that the two expressions differ only in sign. He imaginatively explains these formulas in terms of the integers A and B. Essentially, Bombelli provided further factual evidence to support the idea that any real number may be represented by a complex number. In this sense, Bombelli's idea was a mathematical innovation that added to the depth of set theory's growth. Bombelli acknowledges that none of the three origins. Bombelli went on to demonstrate in 1560 that a real number might be created by manipulating the fictitious phrase provided by Cardano's method [1-18]. Bombelli contends that the two phrases differ only in sign. He imaginatively explains these formulas in terms of the integers A and B. Essentially, Bombelli provided further factual evidence to support the idea that any real number may be represented by a complex number. In this sense, Bombelli's idea was a mathematical innovation that added to the depth of set theory's growth. The suggestion of Leibniz is kept, since $\sqrt{10}$ is real. In fact, the cogency of this depends on the nature of the conjugate, not the specific numbers used. Actually, Leibniz's assertion can be verified by using algebraic method and any two positive integers called a and b. Here's a proof for the statement: Assume the expression is: $\sqrt{a + \sqrt{-b}} + \sqrt{a - \sqrt{-b}}$. Then the expression is squared: $(\sqrt{a + \sqrt{-b}} + \sqrt{a - \sqrt{-b}})^2 = a + \sqrt{-b} + 2\sqrt{a + \sqrt{-b}}\sqrt{a - \sqrt{-b}} + a - \sqrt{-b} = 2a + 2\sqrt{a^2 + b}$. Thus, it has: $\sqrt{a + \sqrt{-b}} + \sqrt{a - \sqrt{-b}} = \sqrt{2a + 2\sqrt{a^2 + b}}$. Then Leibniz was enlightened by Bombelli's L'Algebra when studying cubics. He tried to demonstrate that Cardano's formula is valid without being redefined. He found that the roots of cubic equations can be classified into two types, which are "three real roots" and "two complex roots and one real root". This was thereafter used into the "basic laws of algebra" to improve its validity. However, Leibniz's use of the word "imaginary" was his greatest contribution to the development of the complex. He had a good reason for defining this word, in fact. He claimed, "The gap between being and not-being, which we call the hypothetical root of negative oneness, provided the Divine Spirit with a glorious expression in the miracle of analysis." Even though Leibniz began to employ these numbers, he was unsure of their nature. Consequently, the absence of evidence for complex numbers' geometrical validity prevents them from being classified as actual numbers. Leonhard Euler, one of the greatest mathematical minds in history, created the complex system-one of the most staple and fundamental term about complex numbers. He stated that $i = \sqrt{-1}$, which represented the square root of negative numbers. Euler said that the symbol I is the major determinant that indicates an equation cannot be really solved. Although Euler provided the definition of i, he still held the opinion that complex numbers didn't actually exist. As many mathematicians kept studying complex numbers in the 18th century, some of them consider that there were different types of complex numbers, just like real numbers including rational and irrational ones. However, in a book published by D'Alembert in 1747, he represented that all the complex numbers have a general form: $a + bi$. In fact, complex numbers is a paradoxical term which are infinite like real numbers but can only be represented in on single form [1-18].

2. Complex Number and its topology

The complex numbers denoted by C are the elements that satisfy conditions: Each real number is complex number; There exists an element i such that $i^2 = -1$; Each complex number has its unique representation in cartesian form of $a + bi$ where $a, b \in \mathbb{R}$. The addition and multiplication of complex number holds for commutativity, associativity and distributivity laws. As for complex number $a \in \mathbb{C}$

has its addition inverse: $-a$ such that $a+(-a) = 0$ and multiplication inverse $1/a$ such that $a*(1/a) = 1$ when $a \neq 0$. Representation of complex number are as follows Cartesian Form $z = a + bi$ where $a, b \in \mathbb{R}$ and Polar form $z = \mu e^{i\theta} = \mu(\cos\theta + i\sin\theta)$ $\mu > 0$, and $\theta \in \mathbb{R}$ as the argument of complex number z . The conjugate of complex number $z = m + ni$ denotes as $\bar{z} = m - ni$ for $m, n \in \mathbb{R}$. Properties: Let $m, n \in \mathbb{C}$, then $\overline{mn} = \bar{m}\bar{n}$ $\overline{m+n} = \bar{m} + \bar{n}$, $\overline{\bar{m}} = m$.

The modulus of complex number $z = m + ni$ defined as $|z| = \sqrt{m^2 + n^2}$ for $m, n \in \mathbb{R}$.

Properties are as follow: As for the complex number $m, n \in \mathbb{C}$, it has: $|mn| = |m| |n|$; $|m+n| \leq |m| + |n|$ (triangle inequality). Take $z = 1$ and $n \in \mathbb{Z}^+$ fixed, there are exactly n different numbers such that $w^n = z = 1$.

Prove: Let $\theta = \frac{2\pi}{n}$ $w_1 = 1 * e^{i\frac{2\pi}{n}} = e^{i\theta}$ $w_2 = 1 * e^{i\frac{4\pi}{n}} = e^{2i\theta}, \dots, w_n = 1 * e^{i2\pi} = 1$ are all satisfy $w_i^n = 1$ for $i = 1 \dots n$.

As for disc $D_r(x_0)$ which centered at $x_0 \in \mathbb{C}$ with radius $r > 0$ is said to be open disc if the absolute values of each element on the set of $D_r(x_0)$ to the centre point x_0 is strictly less than r . It can be represented as: $D_r(x_0) = \{x \in \mathbb{C} : |x - x_0| < r\}$. As for disc $D_r(x_0)$ which centered at $x_0 \in \mathbb{C}$ with radius $r > 0$ is said to be closed disc if the absolute values of each element on the set of $D_r(x_0)$ to the center point x_0 is less than or equal to r . It can be represented as: $D_r(x_0) = \{x \in \mathbb{C} : |x - x_0| \leq r\}$. The point x is said to be the boundary point of the set $U \subseteq \mathbb{C}$ when for any disc with center point at x has elements in U and elements not in set U . It can be represented as: $\{x \in \mathbb{C} | \forall D(x, -) \cup U \neq \emptyset \text{ and } \forall D(x, -) \cup \bar{U} \neq \emptyset\}$. The set $\tau \subseteq \mathbb{C}$ is said to be open disc if for every element in τ there exists $r > 0$ such that $D(x, r) \subseteq \tau$. The set $\tau \subseteq \mathbb{C}$ is called closed set if $\mathbb{C} \setminus \tau$ is open in \mathbb{C} .

OR The set $\tau \subseteq \mathbb{C}$ is called closed set if τ contains all its boundary points. The set $\tau \subseteq \mathbb{C}$ is called bounded set if there exist a complex number $z \in \mathbb{C}$ such that $|x| < z$ for every x in the set τ .

The set $\tau \subseteq \mathbb{C}$ is called compact set if set τ is closed and bounded set. For a sequences $r = \{r_1, r_2, \dots\}$ where $r_i \in \mathbb{C}$ is convergence to z when $\lim_{n \rightarrow \infty} |r_n - z| = 0$, which indicates that $z = \lim_{n \rightarrow \infty} r_n$. The set of complex numbers \mathbb{C} is complete. The set $\tau \subseteq \mathbb{C}$ is bounded and its diameter is given by: $\text{diam}(\tau) = \sup_{z_i, z_j \in \tau} |z_i - z_j|$. There are nonempty compact sets in complex plane $\gamma_1,$

$\gamma_2, \gamma_3 \dots$ such that $\tau_1 \supset \tau_2 \supset \tau_3 \supset \dots$ with $\text{diam}(\tau_n) \rightarrow 0$ as $n \rightarrow \infty$, then there exists unique point z such that $z \in \tau$ for all $i \in \mathbb{Z}^+$.

3. Proof of Cauchy's theorem in Disc (Local)

The holomorphic function f defined on open set U . The integral along closed triangle path $\gamma \subset U$ as: $\int_{\gamma} f = 0$. Decompose the rectangle path $\gamma^{(0)}$ into 4 triangles as $\gamma_1^{(1)}, \gamma_2^{(1)}, \gamma_3^{(1)}, \gamma_4^{(1)}$ by bisecting each side and its length $l^{(0)}$ and width $w^{(0)}$. Then it is obvious that: $\int_{\gamma} f = \sum_{i=1}^4 \int_{\gamma_i^{(1)}} f$.

By triangle inequality $\int_{\gamma} |f| \leq \sum_{i=1}^4 \int_{\gamma_i^{(1)}} |f|$, Consequently, there exists $j \in \{1,2,3,4\}$ such that:

$|\int_{\gamma_j^{(1)}} f| \geq \frac{1}{4} \int_{\gamma} |f|$. Then, it can be decomposed into the specific rectangle $\gamma_j^{(1)}$ into 4 rectangles as $\gamma_1^{(2)}, \gamma_2^{(2)}, \gamma_3^{(2)}, \gamma_4^{(2)}$ and its diameter $d^{(1)}$ and perimeter $p^{(1)}$ by bisecting each side and so on. Iteratively, the conclusion would be: $\int_{\gamma} |f| \leq 4^n |\int_{\gamma_j^{(n)}} f|$ and

$l^{(0)} = 2^n l^{(n)}, w^{(0)} = 2^n w^{(n)}$ after decomposing n times. By as for $\gamma^{(0)} \supset \gamma^{(1)} \supset \gamma^{(2)} \supset \dots$ and its perimeter is approximate to 0, there exist a unique point z such that $z \in \gamma_i$ for all $i \in \mathbb{Z}^+$. Since the function f is holomorphic to open set U which contain point z , then $f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)\sigma(z)$ as $\sigma(z) \rightarrow 0$ when $z \rightarrow z_0$

Integrate both parts along path $\gamma_j^{(n)}$: $\int_{\gamma_j^{(n)}} f(z) = \int_{\gamma_j^{(n)}} f(z_0) + f'(z_0) \int_{\gamma_j^{(n)}} (z - z_0) + \int_{\gamma_j^{(n)}} (z - z_0)\sigma(z)$. As $f(z_0)$ and $(z - z_0)f'(z_0)$ are constant and linear function separately, thus $\int_{\gamma_j^{(n)}} f(z_0) = 0$ and $f'(z_0) \int_{\gamma_j^{(n)}} (z - z_0) = 0$. So that: $\int_{\gamma_j^{(n)}} f(z) = \int_{\gamma_j^{(n)}} (z - z_0)\sigma(z)$. Combine conclusion above: $\frac{1}{4^n} \int_{\gamma} |f| \leq |\int_{\gamma_j^{(n)}} f(z)| \leq |\int_{\gamma_j^{(n)}} (z - z_0)\sigma(z)|$. By $(z - z_0) \leq \text{diam}(\gamma_j^{(n)})$, thus $|\int_{\gamma_j^{(n)}} (z - z_0)\sigma(z)| \leq \text{diam}(\gamma_j^{(n)})(2^{-n}l^{(0)})(2^{-n}w^{(0)}) \sup_{z \in \gamma_j^{(n)}} |\sigma(z)|$. Since $\sigma(z) \rightarrow 0$ when $z \rightarrow z_0$, thus $\sup_{z \in \gamma_j^{(n)}} |\sigma(z)| \rightarrow 0$ as $n \rightarrow \infty$. Since $\frac{1}{4^n} \int_{\gamma} |f| \leq |\int_{\gamma_j^{(n)}} (z - z_0)\sigma(z)|$ and $|\int_{\gamma_j^{(n)}} (z - z_0)\sigma(z)| \rightarrow 0$, obviously $\int_{\gamma} f = 0$. Statement of Cauchy theorem on a disc is given as follows. When holomorphic function f defined in an open disc, as for any closed path γ in disc then $\int_{\gamma} f(z)dz = 0$. To prove this, need to justify function f has its primitive. For convenience, denote the disc D with center at origin without loss of generality. With respect to the any path τ in D , divide it into two parts: First from 0 to τ_x point at the horizontal direction then from τ_x to τ_y point at vertical direction, the demonstration of graph as shown below: Denote that $F(z) = \int_{\tau} f(x)dx$ and compare the difference between $F(z+h)$ and $F(z)$ as: $F(z+h) - F(z) = \int_{\tau(z+h)} f(x)dx - \int_{\tau(z)} f(x)dx$. The difference between path of $\tau(z+h)$ and $\tau(z)$ is matrix \vec{u} , thus: $F(z+h) - F(z) = \int_{\tau(z+h)} f(x)dx - \int_{\tau(z)} f(x)dx = \int_{\vec{u}} f(x)dx$. Since function f is holomorphic at point z , the function $f(x)$ as: $f(x) = f(z) + \delta(x)$ where $\delta(x) \rightarrow 0$ as $x \rightarrow z$. As for the equation $F(z+h) - F(z) = \int_{\vec{u}} f(x)dx$, $F(z+h) - F(z) = \int_{\vec{u}} f(x)dx = \int_{\vec{u}} (f(z) + \delta(x)) dx = \int_{\vec{u}} f(z)dx + \int_{\vec{u}} \delta(x) dx$. Moreover, as for any closed path γ in disc, apply the equality: $F(z+h) - F(z) = \int_{\gamma} f(z)dx + \int_{\gamma} \delta(x) dx$, Since $\int_{\gamma} f(z)dx = f(z) \int_{\gamma} dx = h * f(z)$, and also $\int_{\gamma} \delta(x) dx \leq \sup_{t \in \gamma} |\delta(t)| |h|$ which $\lim_{h \rightarrow 0} \sup_{t \in \gamma} |\delta(t)| \rightarrow 0$. Therefore, $F(z+h) - F(z) = \int_{\gamma} f(z)dx + \int_{\gamma} \delta(x) dx = h * f(z)$. The equation can be rewritten as: $\frac{F(z+h) - F(z)}{h} = f(z)$, thus function f has its primitive. Continue for Cauchy's Theorem, since the holomorphic function f has its primitive defined in an open disc, as for any closed path γ in disc, so that $\int_{\gamma} f(z)dz = 0$. As for the holomorphic function f in open set along the path of circle γ with its interior, the integral: $\int_{\gamma} f(z)dz = 0$. Two closed continuous paths α, β in open set U such that they are closed together at same interval $[m, n]$. As for any holomorphic function on U , $\int_{\alpha} f = \int_{\beta} f$. The closed continuous path α in open set U is said to homotopic at point n if for any holomorphic function f such that $\int_{\alpha} f = 0$. Two paths α, β in open set U are homotopic then for any holomorphic function u in U , $\int_{\alpha} f = \int_{\beta} f$.

4. Summary

An intriguing and esoteric area of research is complex analysis. Over the past few centuries, mathematicians have produced ground-breaking work in this field. Mathematicians originally defined

complex numbers before further examining their algebraic and geometric characteristics. They just recently learned about and studied sophisticated calculations, after hundreds of years. Holomorphic functions have convergent power series local representations. This remarkable finding, which Cauchy made between 1830 and 1840, contributes to the explanation of the intriguing characteristics of holomorphic functions. One of the key ideas in complex analysis is the Cauchy integral theorem. In this paper, a detailed proof of Cauchy's theorem on a local disc is given. We may talk about the complex analytical method's application to the solution of linear differential equations in the future. Additionally, we may concentrate on Fourier series and attempt to utilize these techniques to solve a problem involving modular forms, a class of analytic functions defined on the upper plane.

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