Charaterization of Holomorphic function using power series

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Abstract. Complex analysis is a subfield of mathematical analysis that focuses on the operations of complex numbers. The theory of functions of a complex variable is another name for it. Holomorphic functions, is the key object studied in the filed of complex analysis. These functions can be differentiated, have negative values, and are defined on the complex plane. Some of the ideas, concepts, and techniques frequently utilized in research include the residue theorem, Cauchy integral formula, Laurent series expansion, etc.In particular, algebraic geometry, fluid dynamics, quantum mechanics, and other related sciences have made substantial use of complex analysis in mathematics, physics, and engineering over the years. Girolamo Cardano and Raphael Bombelli, two Italian mathematicians, initially recognized complex numbers while attempting to solve a particular algebra in the 16th century, while Cauchy and Riemann further improved it in the 19th century.This essay explores the arithmetic features of complex numbers firstly. Secondly, a thorough interpretation of the characteristics of holomorphic functions is provided. Finally, using power series methods, the holomorphic function on the disc is characterized.

Keywords: complex numbers, holomorphic function, power series.

1. Introduction

A branch of mathematical analysis called complex analysis studies the functions of complex numbers. It is also sometimes referred to as the theory of functions of a complex variable. Complex analysis has been used extensively in mathematics, physics, and engineering over the years, particularly in the areas of algebraic geometry, fluid dynamics, quantum mechanics, and other related fields. It dates back to the 16th century, when Italian mathematicians Girolamo Cardano and Raphael Bombelli first noticed complex numbers while attempting to solve a particular algebra, and was later developed by Cauchy and Riemann in the 19th century[1-18]. In the sixteenth century, complex numbers were first studied. Girolamo Cardano discovered complex numbers in 1535 when he used an equation to get the square root of a negative number. Cardano, however, stopped researching this root[1-15]. Over the following 150 years, researchers focused on the distinction between this negative number's root and the typical root and attempted to depict this peculiar root on the coordinate axis. In the disciplines of algebra and geometry, "imaginary number" has now been differentiated for the first time from "real number." Over the following years, the definition of complex numbers was refined. The set of the complex number is x+iy [1-18]. People use this set to represent a number z, which is $z=x+iy \in C$. In this equation, x represents the real point, y represents the imaginary point, and $i=\sqrt{(-1)}$.

Holomorphic functions, is the key object studied in the filed of complex analysis. These functions can be differentiated, have negative values, and are defined on the complex plane. Some of the ideas, concepts, and techniques frequently utilized in research include the residue theorem, Cauchy integral formula, Laurent series expansion, etc. Complex analysis has a wide range of uses and is essential to many other branches of physics and mathematics. encompassing applied mathematics, fluid dynamics, thermodynamics, and electrodynamics. Real function differentiability is easier to verify than complicated function differentiability. A power series, for instance, may be used to represent every open disk of every regular function in its domain.

2. Main works

2.1. Complex Number and Complex Plane

For these three operations, addition and multiplication are relatively simple in real numbers because they along with i^2=-1, and in complex numbers, they are associative and admit identities (0 and 1, respectively), admit inverse -z and $\frac{1}{z}$ (for z$\neq$ 0), and satisfy the distributive law. Give some examples of the distributive law

$$\overline{z+w} = \bar{z} + \bar{w} \tag{1}$$

$$\overline{zw} = \bar{z} \cdot \bar{w} \tag{2}$$

$$\mathrm{Re}(z)=\frac{1}{2}(z+\bar{z}) \tag{3}$$

$$\mathrm{Im}(z)=\frac{1}{2i}(z-\bar{z}) \tag{4}$$

$$|z| = (z\cdot\bar{z})^{\frac{1}{2}} \tag{5}$$

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ is the last identity for all $z, w \in C$ $(z \neq 0)$, which also derives by "conjugating the denominator "

$$\frac{1}{x+iy} = \frac{1}{x+iy}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{\overline{x+\imath y}}{|x+iy|^2} \tag{6}$$

From the identities shown before, it is clearly derive the value of $|zw|^2$

$$|zw|^2 = (zw)\cdot(\overline{zw}) = zw\bar{z}\bar{w} \to |zw| = |z||w| \tag{7}$$

Also, because $z\epsilon C$, if z is real, $\bar{z} = z$, and if z is imaginary, $\bar{z} = -z$. In the coordinate plane, to show the complex number. X-axis and y-axis respectively represent real axis and imaginary axis. Moreover, z is the coordinate point, and x, y in the equation are its coordinate. Thus, coordinate plane is called the complex plane. According to the prior R^2's distance formula, $|z|$ means the distance from z to the origin point. Likewise, $|z-w|$ means the distance from z to w. Additionally, the operations of multiplication and addition are visualized in this way. Especially for multiplication, it needs to be described by polar coordinates and transit for C. For instance

$$\left|\frac{z}{|z|}\right| = |z|\cdot\left|\frac{1}{|z|}\right| = |z|\cdot\frac{1}{\lceil z\rceil} = 1 \tag{8}$$

In the coordinate plane with the circle with an original central point, $\theta \in \mathcal{R}$, so that

$$\frac{z}{|z|} = \cos\theta + i\sin\theta \tag{9}$$

$\frac{z}{|z|}$ can also be written as $e^{i\theta}$. Then, it can be derived to get

$$z = |z|e^{i\theta} \tag{10}$$

This is the polar form of z. The angle $\theta \in \mathcal{R}$ $(0 \leq \theta \leq 2\pi)$ is the argument of z, which is devoted arg z. By the polar form and distributive law,

$$zw = |z|e^{i\cdot argz}\cdot|w|e^{i\cdot argw} = |zw|\cdot e^{i(argz+argw)} \tag{11}$$

Therefore, the distance from z to zw consists of rotating counterclockwise by $arg\ w$ scaling z by

$|w|$. As same as z to iz, $arg\ i = \frac{\pi}{2}$. Above all, complex conjunction

$$\overline{x+\imath y} = x - iy \tag{12}$$

Is the reflection of the real axis. 2×2 matrices can prove the existence of i

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathcal{R}) \tag{13}$$

$$i^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

Thus, a formula is summarized by this

$$\left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + i\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathcal{R}\right\} = \left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y \in \mathcal{R}\right\} \tag{15}$$

Owing to the meaning of absolute value in the coordinate plane, there is a triangle inequality $|z+w| \leq |z| + |w|$. Likewise $||z| - |w|| \leq |z - w|$, accordingly

$$|Re(z)| \leq |z| \text{ and } |Im(z)| \leq |z| \tag{16}$$

The sequence $(z_n)_{n\in w} \subset C$ converges to $w \in C$, if $\lim_{n\to\infty} |z_n - w| = 0$, in which case, it writes

$$\lim_{n\to\infty} z_n = w \tag{17}$$

And w is called the limit of the sequence. In other words, $(z_n)_{n\in w} \subset C$ converges to w if $\forall \varepsilon > 0\ \exists\ N \in N$ so $\forall n \geq N\ |z_n - w| < \mathcal{E}$ because

$$|Re(z)|, |Im(z)| \leq |z| = (Re(z)^2 + Im(z)^2)^{\frac{1}{2}} \tag{18}$$

Then

$$\lim_{n\to\infty} z_n \Leftrightarrow \lim_{n\to\infty} Re(z_n) = w \ and\ \lim_{n\to\infty} Im(z_n) = w \tag{19}$$

There is an important sequence—Cauchy sequence. The particular formula of this sequence is

$$|z_n - z_m| < \varepsilon \tag{20}$$

Cauchy is a convergent sequence. Since the term is close to the limit and eventually each other, it turns out the converse is true. By the Cauchy sequence converge, complex numbers are complete. If $(Zn)_{n\in N} \epsilon C$ as the Cauchy sequence, from $|Re(z)|, |Im(z)| \leq |z|$, $(Re(z_n))_{n\in N}, (Im(z_n)_{n\in N}) \in R$ are also Cauchy sequences. At this point, real numbers are complete, and both imaginary and real parts, so it defines $w=x+iy$. Then,

$$\lim_{n\to\infty} |z_n - w| = \lim_{n\to\infty} ((Re(z_n) - x)^2 + (Im(z_n) - y)^2)^{\frac{1}{2}} = 0 \tag{21}$$

Above this, $(Zn)_{n\in N} \epsilon C$ converge to w.

The open disc of radius r centered at z_0 is

$$Dr(z_0) \coloneqq \{z \in C | |z - z_0| < r\} \tag{22}$$

for $z_0 \in C\ and\ r > 0$, but the close disc is

$$\overline{Dr} z_0 = \{z \in C | |z - z_0| \leq r\} \tag{23}$$

And the circle of the radius r centered at z_0 is

$$Cr(z_0) \coloneqq \{z \in C | |z - z_0| = r\} \tag{24}$$

For the topological properties, r can identify complex plane, and it has same notions of convergence and distance, but the limitation is that those properties are transaction from r to complex number. $\Omega \subset C$ is opened. Follow this, if all $z \in \Omega$ that r>0, then

$$Dr(z) < \Omega \tag{25}$$

However, if Ω's complement $\Omega^2 = \frac{C}{\Omega}$, Ω is closed. There are some important points to remember. The first one is that both C and ϕ are open and closed. Next one is that

$\{x + iy \in C | x \geq 0 \; and \; y > 0\}$ is neither open nor closed. The last one is that $Dr(z_0)$is open and $\overline{Dr}(z_0)$ is close, and any $z \in C \; and \; r > 0$. Then, when $z \in Dr(z_0)$, $D_{r-|z-z_0|}(z) < Dr(z_0)$, and when $z \in \overline{Dr}(z_0)$, $D_{r-|z-z_0|}(z) < \overline{Dr}{z_0}^0$. Thus, open or closed is not a dichotomy, and sets can be open, close, either, or neither. It can also check if the sets is open or not by the definition. One of the ways to check it is that if any convergent sequence $z_{n_{n\in\omega}} < \Omega$ one has $\lim_{n\to\infty} z_n \in \Omega$, then the set $\Omega < C$. This is because that if set Ω is closed and $z_{n_{n\in N}} \subset \Omega$ be a convergent sequence. In $\omega \coloneqq \lim_{n\to\infty} z_n$ (r>0), because Ω^{C} is open, if $\omega \notin \Omega$, $Dr(\omega) \subset \Omega^{\mathsf{C}}$. Let $N \in \mathbb{N}$ be such that for $n \geq N, |z_n - \omega|$. Then, form $z_n \in Dr\,(\omega) \subset \Omega^{\mathsf{C}}$ to $z_n \subset \Omega$. Therefore, contradicting $\omega \in \Omega$.

Moreover, some important closed and open sets are combined to $\Omega \subset C$. In this formula, $z \in C$ is interior point of Ω that r>0, then $Dr(z) \subset \Omega$. The set of all interior points of Ω is denoted Ω. Also, if $z_{n_{n\subseteq N}} \subset \frac{\Omega}{\{z\}}$ with $\lim_{n\to\infty} z_n = z$, z is limit point of Ω. The closure of Ω is also its union, limit point, and is denoted $\overline{\Omega}$. Lastly, the boundary of Ω is $\partial\Omega \coloneqq \frac{\bar{\Omega}}{\Omega^o}$.

$\Omega \subset \mathrm{C}$ is bonded is based on $\Omega \subset D_R(o)$, if $|z| \leq R$(R>0) for all $z \in \Omega$. If Ω is bonded, the diameter shows that $diam(\Omega) \coloneqq \sup_{z,w\in\Omega} |z - w|$ which can also be said it is compact if Ω is either closed or bonded. This might be different in different definitions, but in complex plane, they are equivalent. In another situation of $\Omega \subset \mathrm{C}$ is an open cover, which is $\{Ui \subset \Omega | i \in I\}$ of open sets satisfying

$$\bigcup_{i\in I} Ui \supset \Omega \tag{26}$$

A subcover is a subcollection, $\{U_i | i \exists J, J \subset I\}$ is still covers Ω. Also, in this formula, every sequence $z_{n_{n\in N}} \subset \Omega$ has a subsequence $z_{n_{k\in N}}$ converging to a point Ω, and every open cover $\{Ui \subset C | i \in I\}$ of Ω has a finite subcover $\{Ui_1, Ui_2, \dots, Ui_n\}$. However, $\Omega \subset \mathrm{C}$ can be disconnected with open sets $U_1, U_2 \subset C$, which respectively satisfying $U_1 \cap U_2 = \phi$. It can also be connected, which is called region in open set in complex number. In $\Omega \subset \mathrm{C}$, if $z, w \in \Omega$, the path from z to w in the Ω is continuous that $f: [z, w] \to \Omega$. Ω is path connected in z and w. therefore, being open is a necessary hypothesis in these properties. The topologists' comb in the above example is connected but not open and one can show it is not path connected.

2.2. Functions on the Complex Plane

Functions are always considered to satisfy continuity or differentiability. As continuous, in $f: \Omega \to C(\varepsilon > 0)$ when $\Omega < C$ is continuous at $z_0 \in \Omega$. Also, when $z \in \Omega$ satisfies $|z - z_0| < \delta$, $\delta > 0$ has existed. Then, there is $|f(z) - f(z_0)| < \varepsilon$, which f is continuous on Ω if it is continuous at every $z_0 \in \Omega$ either. In the same way, f is continuous to z_0 if for any sequence $(z_n)_{n\in N} < \Omega$ converging to z_0, the sequence $(f(z_0))_{n\in N}$. As maximum or minimum, in $f: \Omega \to C$, f attains maximum or minimum at $z_0 \in \Omega$ if $|f(z)| \geq |f(z_0)| or \, |f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. The theorem is a useful method to analyze the features of compact sets.

A continuous function can be bonded and attained extreme values on a compact set $\Omega < C$. Let $\Omega < C$ be an open set in function $f: \Omega \to C$, then f is holomorphic at $z_0 \in \Omega$ if

$$\lim_{C\ni h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \tag{27}$$

exists. In addition, if $\omega \in C$ exists, for all $\varepsilon > 0$, $\delta > 0$ has existed. So whenever $h \in \backslash\{0\}$ satisfies $|h| < \delta$ then one has $z_0 + h \in \Omega$ and

$$\left| \frac{f(z_0+h)-f(z_0)}{h} - \omega \right| < \varepsilon \tag{28}$$

In this case, w is the derivative of f at z_0 and write $f'(z_0) \coloneqq w$. From this formula, if f is holomorphic on Ω, then all of f are holomorphic at every $z_0 \in \Omega$. f is entire holomorphic when it is on C, and a function on a closed subset $\vee\subset \Omega$ either, if it is holomorphic on some open set containing V. Definition of differentiability for real-valued functions relates to this theorem. By contrast, the definition which is mentioned is more explicitly because if f is holomorphic on an open set then it is infinitely differentiable there. Thus, this is because h is allowed to approach 0 from any path whereas for functions on R which has $h \downarrow 0$ and $h \uparrow 0$. For example, $f(z) = z$ is entire and its derivative is I, then

$$|\frac{f(z_0+h)-f(z_0)}{h}| = |\frac{z_0+h-z_0}{h} - 1| = 0 \tag{29}$$

Moreover, for any $a_0, \dots, a_n \in C$, the polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is entire with $p'(z) = a_1 + \cdots + n a_n z^{n-1}$. This follows from $f'(z) = I$.

To satisfy that $f: \Omega \to C$ is holomorphic, only if there exists $a \in \Omega$ and $r > 0$ so that for $|h| < r$

$$f(z_0 + h) = f(z_0) + ah + h\psi(h) \tag{30}$$

Where $\psi: Dr(0)\{0\} \to C$ satisfies $\psi(h) \to 0$ as $\text{h} \to 0$. If $a = f'(z_0)$ in this case and choose $r > 0$ so that $Dr(z) < \Omega$, and then there is a set

$$\psi(h) := \frac{f(z_0+h)-f(z_0)}{h} - f'(z_0) \tag{31}$$

Because $h \in Dr(0)\backslash\{0\}$, $\lim_{h\to 0} \psi(h) = 0$ become holomorphic at z_0 obey the definition of f. Let $0 < \delta < r$ be such that $|\psi(h)| < \varepsilon$ for $|h| < \delta$ by given $\varepsilon > 0$, the $|h| < \delta$ implies

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - a\right| = |\psi(h)| < \varepsilon \tag{32}$$

This also means that if it is holomorphic at a point, which is also continuity at the same point

$$|f(z_0 + h) - f(z_0)| = |f'^{(z_0)} + \psi(h)| \cdot |h|\overrightarrow{h \to 0}0 \tag{33}$$

If f and g are holomorphic on Ω, then $f + g$ is holomorphic on Ω with $(f + g)' = f' + g'$, $f \cdot g$ is holomorphic on Ω with $(fg)' = f'g + fg'$, and ${}^{f}/_{g}$ is holomorphic on ${}^{\Omega}/_{\{z \in \Omega: g(z) = 0\}}$ with $(\frac{f}{g}) = \frac{f'g-fg\prime}{g^2}$. Moreover, if $f: \Omega \to U$ and $g: \text{U} \to C$ are holomorphic, then $g \circ f$ is holomorphic on Ω with

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \tag{34}$$

In Cauchy-Riemann Equations, $f: \Omega \to C$ always define real-valued functions

$$u(x, y) := Re(f(x + iy)) \tag{35}$$

$$v(x, y) = Im(f(x - iy)) \tag{36}$$

Then from the original formula, $f(x + iy) = u(x, y) + iv(x, y)$. Let f is holomorphic at $z_0 = x_0 + iy_0$, and $f(x, y) := f(x + iy)$ as a function on R^2, then for its partial derivatives

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{R \ni h \to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \tag{37}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{R \ni h \to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \tag{38}$$

Make sense. $f'(z_0)$ relates to this by considering different paths $C \ni h \to 0$. First to consider $R \ni h \to 0$

$$f'(z_0) = \lim_{R \ni h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{R \ni h \to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \frac{\partial f}{\partial x}(x_0, y_0) \tag{39}$$

Then, after considering $iR \ni ih \to 0$,

$$f'(z_0) = \lim_{iR \ni h \to 0} \frac{f(z_0+ih)-f(z_0)}{h} = \frac{1}{i} \lim_{iR \ni h \to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \frac{1}{i}\frac{\partial f}{\partial x}(x_0, y_0) \tag{40}$$

Ultimately,

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{1}{i}\frac{\partial f}{\partial x}(x_0, y_0) \quad (41)$$

From the complex number, $f = u + iv$ included real and imaginary parts, which also obtains the Cauchy-Riemann equations that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)\ and\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \quad (42)$$

Therefore, if f is holomorphic on an open set $\Omega \subset C$, its real and imaginary parts satisfy all partial derivates on the Cauchy-Riemann equation. Also, because holomorphic functions are infinitely differentiable, their partial derivatives will be continuous. Even the converse of this is valid. Let $\Omega < R^2$ be an open subset, and suppose $u, v\colon \Omega \to \mathrm{R}$ have continuous partial derivatives that satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ and\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (43)$$

on Ω. Then $f = u + iv$ is holomorphic on Ω with

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}(x_0, y_0) - i\frac{\partial v}{\partial x}(x_0, y_0) \quad (44)$$

Power series centered at $z_0 \in C$ is a function of the form $f(z) = \sum_{n=0} a_n(z - z_0)^n$ where $(a_n)_{n\in N} \subset C$. Power series converges absolutely at z if

$$\sum_{n=0} |a_n||z - z_0|^n < \infty \quad (45)$$

The domain of power series is the set of its absolutely converges points. Ther e are examples of holomorphic functions from power series. In these cases, every holomorphic function is locally equal to the power series. If the power series converges absolutely at z, then the partial sums become the Cauchy sequence: let $N_0 \in N$ satisfies $\sum_{n=N_0}^{\infty} |a_n||z - z_0|^n < \varepsilon$ for $\varepsilon > 0$, then for $M > N \geq N_0$

$$\left|\sum_{n=0}^{M} a_n(z - z_0)^n - |\sum_{n=0}^{N} a_n(z - z_0)^n\right| \leq \sum_{n=N+1}^{M} |a_n||z - z_0|^n \leq \sum_{n=N_0}^{\infty} |a_n||z - z_0|^n < \varepsilon$$

Thus, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ defines to the limit of this Cauchy Sequence, which exists by the completeness of C. Moreover, the absolute convergence at z depends on $r := |z - z_0|$, so the domain of the power series contains z_0.

2.3. Radius of Converge

The supremum of R is the radius of convergence of the power series. By the definition, when $|z - z_0| < R$, the power series converges absolutely at z, and when $|z - z_0| > R$, it diverges. For $|z - z_0| = R$, the series may converge absolutely, diverge, or only conditionally converge on a subset of the circle $C_R(z_0)$.

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \quad (46)$$

is a useful formula for computing R, which gives the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. To prove this, let L be the limsup in the statement to prove 0<L<∞. Given $|z - z_0\ | < \frac{1}{L}$, let $\varepsilon > 0$ to small enough then

$$r := (L + \varepsilon|z - z_0|) < 1 \quad (47)$$

By the definition, $|a_n|^{\frac{1}{a}} \leq L + \varepsilon$ for all sufficiently large n, then

$$|a_n||z - z_0|^n \leq (L + \varepsilon)^n \cdot |z - z_0|^n = r^n \quad (48)$$

Thus, $\sum|a_n||z-z_0|^n$ converges by comparison with the geometric series $\sum r^n = \frac{1}{1-r}$ shows $\frac{1}{L} \leq R$ the radius of convergence. Next, if $|z-z_0| > \frac{1}{L}$, choose $\varepsilon > 0$ so that

$$s := (L-\varepsilon)|z-z_0| > 1 \tag{49}$$

Then, for each $n \in N$ there exists $k \geq n$ so that $|a_n|^{\frac{1}{k}} \geq L-\varepsilon$. In this case,

$$|a_n||z-z_0|^n \geq (L+\varepsilon)^k \cdot |z-z_0|^k = S^k > 1 \tag{50}$$

Thus, the series $\sum|a_n||z-z_0|^n$ diverges since there are infinitely many terms greater than I . thus, we cannot have $\frac{1}{L} < R$ and $\frac{1}{L} = R$ as claimed.

2.4. f becomes differentiable on $D_R(z)$

As a general phenomenon, let $f(z) = \sum_{n=0}^{M} a_n(z-z_0)^n$ be a power series with radius of convergence $R > 0$. Then f is holomorphic on $D_R(z_0)$ with

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1} \tag{51}$$

Moreover, f' also has radius of convergence R. It shows that a function analytic on Ω is holomorphic on Ω.

Let $f(z) = \sum_{n=0}^{M} a_n(z-z_0)^n$ be a power series with radius of convergence $R > 0$. Then f is infinitely differentiable on $D_R(z)$ with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z-z_0)^{n-k} \tag{52}$$

Moreover, each $f^{(k)}$ has radius of convergence R. For $f: \Omega \to C$, f is analytic at $z_0 \in C$, if there exists a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with positive radius of convergence $R > 0$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{53}$$

For all z in a neighborhood of z_0. In additional, if f is analytic at all $z_0 \in C$, it is analytic on Ω.

3. Summary

This essay examines complex analysis from three perspectives: the integral of a complex function, the integral of a complex line, and Cauchy's theorem. In this work, specific theorems are provided together with proofs. This study provides background information on complicated analysis before exploring this subject. The historical information and the general introduction of complex analysis are combined to demonstrate the relevance of complex analysis in this work. Following that, certain rules and theorems are employed to define complex numbers. Considering that complex numbers are a key concept used in complicated analysis. In this paper, firstly arithemetic properties of complex numbers are explored. Secondly, the properties of holomorphic functions are interpreted in detail. At last, charaterization of holomorphic function on the disc is given using techniques of power series.

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