The additional proof and application of Cauchy-Riemann Equation

Guangxun Zhang*
Beijing Luhe international academy, Beijing, China
*Corresponding author: dsoule1@une.edu

Abstract. As far back as the seventeenth century, Newton and Leibniz introduced and refined the idea of calculus, and the concepts of derivative and differentiation gradually entered the mathematical world as powerful tools. However, after the appearance of the theory of complex functions in the 18th century, the conventional methods used to analyze the properties of complex functions were somewhat limited. Thus, mathematicians Cauchy and Riemann came up with the idea of the "C-R equation" and provided the necessary and sufficient conditions in derivable complex functions based on the research done by d'Alembert. This equation laid the foundation for the analytic function. This paper introduces the idea of the limit of two variables function and shows both the concept of and graphical meaning partial derivative, paving way for the concept of complex function and that of C-R equation. In addition, the paper specifically proves the C-R equation by a much more rigorous method with the routes from the linear equation and applies this important equation to determine the number of and the locations of the differentiable points in a complex plane. It is also one of the factors to determine the analyticity of complex functions. At the end of the paper, an example will be provided to illustrate the application of the C-R equation.

Keywords: Limit, Partial Derivative, Complex Analysis, Cauchy-Riemann Equation.

1. Introduction

The theory of simple complex functions is one of the most unique mathematical creations developed in the 19th century. It was once considered one of the most harmonious theories in science, and its ideas profoundly influenced twentieth-century mathematics. The complex function theory takes analytical functions as the research object. The main research object of mathematical analysis is the real function whose value is the real variable. However, complex functions are self-varying functions with complex numbers, so there are differences between the real function (i.e., the function of one or two variables) and the complex function [1, 2]. For instance, in complex functions, the differential mean value theorem doesn't exist [3]. Therefore, the Cauchy-Riemann equation is an important part of the complex function theory because it deeply reveals the "structure" of analytical functions when the real and imaginary parts of analytical functions are differentiable and satisfy the Cauchy-Riemann condition [4].

In addition, in four-dimensional complex functions, some of the traditional methods (i.e., the way to determine the differential in the two-dimensional plane) in unary function seem to be useless in the complex plane, as the limits in the complex function can be traced by different routes [5, 6]. Although it is possible to use the definition of an analytic function to determine its analyticity, this is a very complicated process. However, the Cauchy-Riemann equation gives a much easier way to determine the differentiability of complex functions and is an extremely important formula for complex analysis.

Even though the C-R equation can be proved in a much easier way, which is usually shown in textbooks, this method only considers two special cases, which are both from X-axis and Y-axis [7]. Thus, this paper will use a more comprehensive method and show the feasibility of the C-R equation when it comes to different paths followed by any linear function (i.e., $y = kx$) on a complex plane. Since the complex function can be divided into two different functions of two variables (one is the real part $v(x, y)$, and another is the imaginary part $u(x, y)$), the way determining the differentiability of complex function, to some extent, is similar to that of function of two variables [8]. Because the C-R equation can simply analyze the differentiability in complex plane, once the points in the complex plane satisfy the two conditions of C-R equation (this part will be illustrated in 2.3), those points can be seen as “differentiable” [9]. Thus, due to the special properties of the C-R equation, it is much more effective and easier to determine the number of differential points in complex plane and their exact locations. In addition, the C-R equation is also one of the important factors to determine the analyticity of a complex function$f(z)$. As a matter of fact, the C-R equation, as a mathematical tool, contributes a lot to the analytic function, especially to complex analysis.

2. Organization of the Text

2.1. Background information

2.1.1 The limit of the function of two variables

The definition of the limit of a binary function is much more complicated than that of unary function. In a unary function, the dependent variable is one-dimensional and has only two paths, which are either from the positive x-axis or negative x-axis, and once the independent value y from both x_+ and x_- direction is equivalent, the function at that point can be regarded to have “limit” [10].

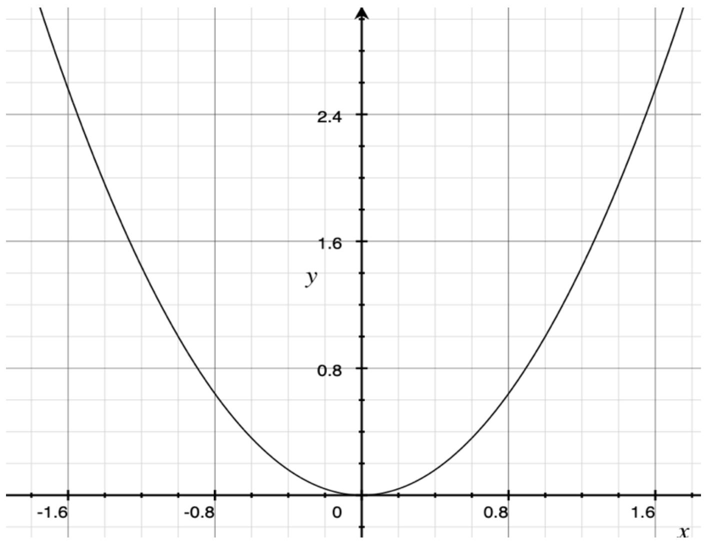

Fig.1 In this case, the function $y = x^2$ has limit everywhere because the value of y is always same from either $x_-$and x_+.

However, when two dependent values are involved in an x-y plane, on which a path can go from a single point to infinity, different paths being chosen may create different values of limits. For example, $p(x, y)$ can approach the fixed point $p(x_0, y_0)$ along any path regardless of its shape (see Figure 2) [11]. However, once all the values from different paths are the same, the function of two variables can be defined to have a “limit”.

2.1.2 The graphic meaning of partial derivative

In unary functions, the derivative is usually used to express the rate of change of the independent variable with respect to the dependent variable. Thus, the graphic meaning of derivative can be seen as a tangent line of the function of one variable at a given point. However, when it comes to $f(x, y)$ with two different independent variables x and y, it seems to be complicated to express. Thus, it is necessary to fix one variable and treat it as a constant, and then do the derivative with respect to one another variable. The geographic meaning of the partial derivative with respect x is the slope of the tangent line along the X-axis at$(x, y, f(x, y))$ on a surface$z = f(x, y)$. Similarly, the geographic meaning of the partial derivative with respect to y is the slope of the tangent line along the Y-axis at $(x, y, f(x, y))$on a surface $z = f(x, y)$ [12]. To be more specific, by paralleling the “fixed variable” axis, the partial derivative is a tangent line on the plane of dependent and a “unfixed variable” [13] (see figure 2).

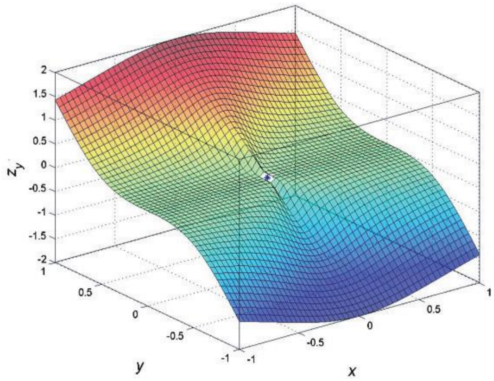

Fig.2 This a a partial derivative with respect to y of a function of two variables $y = x^2 + y^4$ [13].

The concept and methodology of partial derivative will be used to prove C-R equation discussed in part (2.4)

2.2. The graphic concept of complex function

Complex function, whose variable is complex number, consists of two parts, which are real part $u(x, y)$ and imaginary part $v(x, y)$. While the independent variable z is in a two-dimensional plane, complex function is in a totally different complex plane. Thus, by mapping its independent variable, or its domain, in the one complex plane $f(z)$ and its ranges in another complex plane, the graph of complex function can be expressed in two planes (see Figure 3). In addition, the four-dimensional complex function can be expressed in other way: using color to represent its imaginary part (see Figure 4) in three-dimensional curve.

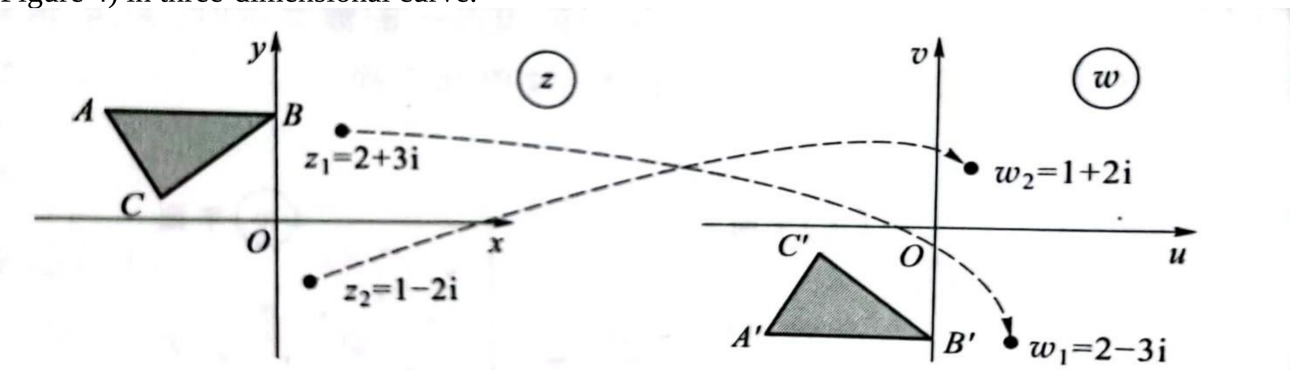

Fig.3 The left graph is the domain of complex function and maps to the right graph, the range of complex function, by $w = \bar{z}$, which means that the original graph will be symmetric to $x - axis$ [14].

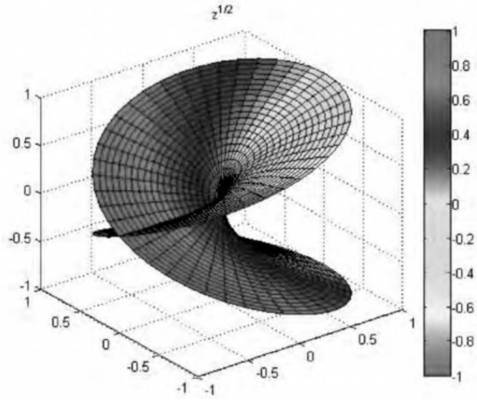

Fig.4 This is four-dimensional graph of $y = \sqrt{z}$, where color represents Imaginary part [15].

In addition, the complex function also has a special property---analyticity. Analyticity means that if the function $w = f(z)$is differentiable and satisfies the C-R equation in region D, then $f(z)$ is regarded to be an analytic function (or holomorphic function, regular function) in region D.

2.3. The concept and importance of C-R equation

The Cauchy-Riemann condition plays an important role in complex functions and its application in complex function theory can be reflected in many important aspects, one of which is the search for differentiable points in complex functions. By utilizing this special property, the Cauchy-Riemann condition can be used to judge the analyticity of a complex function $f(z)$ easily when the $f(z)$ is continuous (the limit of certain point z_0 is equal to the actual value). [16]

2.4. The proof of C-R equation

C-R equation can be proved by different routes being chosen, but the traditional, basic proof is to make the limit go along the x-axis and y-axis.[17] Since the variable of complex function can be written in the form $z = x + iy$. Thus, the complex function can be expressed as:

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

Where both $u(x, y)$and $v(x, y)$are two variables' functions
Based on the formula of derivative,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ where } \Delta z = \Delta x + i\Delta y \tag{2}$$

Thus, the derivative of complex function can be written as:

$$\frac{dy}{dz} = \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y} \tag{3}$$

It is very complicated to handle variable Δx and Δy at same time, so the the paths can be chosen from either horizontal direction, the x-axis, and vertical direction, the y-axis.

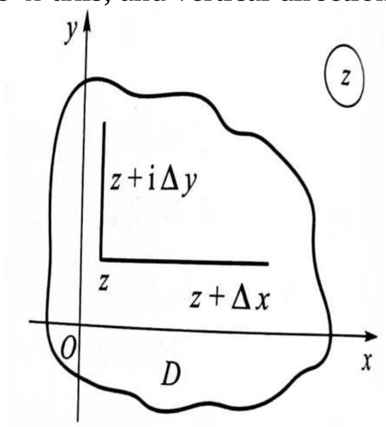

Fig.5 the limits from positive x-axis and positive y-axes converge into one point z.

Horizontal direction from $x - axis$:
Since the route starts from $(\Delta x, 0) \to (0,0)$, where Δy stays constant.
That is:

$$\frac{dy}{dz} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x} \tag{4}$$

Based on the rule of partial derivative in (2.1.2):

$$\frac{dy}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{5}$$

Vertical direction from $y - axis$:
Since the route starts from $(0, \Delta y) \to (0,0)$, Δy stays constant.

That is:

$$\frac{dy}{dz} = \lim_{\Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} i \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{i\Delta y} \tag{6}$$

Since $\frac{1}{i} = -1$, the equation can also be written as:

$$\frac{dy}{dz} = \lim_{\Delta y \to 0} \frac{v(x,\ y+\Delta y) - v(x,\ y)}{\Delta y} - i \lim_{\Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta y} \tag{7}$$

Based on the rule of partial derivative:

$$\frac{dy}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{8}$$

Thus, it is easy to know the two conditions of C-R equation are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{9}$$

By following the two “special” routes from either $x - axis\ and\ y - axis$, it is easier to get the result of C-R equation.

However, it is not rigorous to prove and conclude the results of the C-R equation based on just two “special” routes because the limit can follow the different routes from any direction and with varied shapes. Thus, proving the feasibility of the C-R equation by taking different routes regardless of paths’ shapes is much convincing, and the following part will prove the C-R equation by two linear functions along which the limit goes. (e.g., $y = k_1x$ and $y = k_2x$, where $k_1 \neq k_2$).

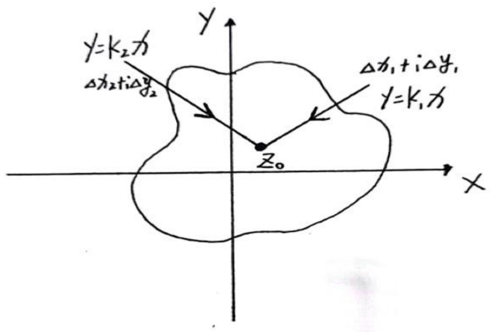

Fig.6 the limit of point z_0 is come by two linear functions, which are $y = k_1x\ and\ y = k_2x$

Due to the fact the $y = k_1x$, $\Delta y = k_1\Delta x$. If $\Delta x \to 0$, it is also true that $\Delta y \to 0$. This satisfies the condition of derivative, which is $\Delta x + i\Delta y = \Delta z \to 0$.

Thus, the derivative can be written as a form of:

$$\frac{dy}{dz} = \lim_{\substack{\Delta y \to 0 \\ \Delta y = k_1\Delta x}} \left(\left(\frac{u(x_0+\Delta x,\ y_0+\Delta y) - u(x_0, y_0)}{\Delta x + ik_1\Delta x}\right) + i\left(\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + ik_1\Delta x}\right)\right) \tag{10}$$

Then, separate the real part and imaginary part.

For real part, it can be written as:

$$\lim_{\substack{\Delta y \to 0 \\ \Delta y = k_1\Delta x}} \left(\frac{u(x_0+\Delta x,\ y_0+\Delta y) - u(x_0, y_0)}{\Delta x + ik_1\Delta x}\right) \tag{11}$$

At first glance, it is hard to handle formula (9) containing two variables. However, the breakthrough here is to utilize $u(x_0,\ y_0+\Delta\mathrm{y})$ and put it on the numerator of formula (9).

Sorting this out,

$$\lim_{\substack{\Delta y\to 0\\ \Delta y=k_1\Delta x}}\left(\frac{u(x_0+\Delta\mathrm{x},\ y_0+\Delta\mathrm{y})-u(x_0,y_0)+u(x_0,y_0+\Delta\mathrm{y})-u(x_0,y_0+\Delta\mathrm{y})}{\Delta x+ik_1\Delta x}\right) \tag{12}$$

Thus, by doing so, the real part of derivative of $f(z)$ can also be split into two parts, each of which is partial derivative with respective to $x\ and\ y$.

One is:

$$\lim_{\substack{\Delta y\to 0\\ \Delta y=k_1\Delta x}}\left(\frac{u(x_0+\Delta\mathrm{x},\ y_0+\Delta\mathrm{y})-u(x_0,y_0+\Delta\mathrm{y})}{\Delta x+ik_1\Delta x}\right) \tag{13}$$

Another one is:

$$\lim_{\substack{\Delta y\to 0\\ \Delta y=k_1\Delta x}}\left(\frac{u(x_0,\ y_0+\Delta\mathrm{y})-u(x_0,y_0)}{\Delta x+ik_1\Delta x}\right) \tag{14}$$

Take the $\frac{1}{1+ik_1}$off from formula (11):

$$\frac{1}{1+ik_1}\left(\lim_{\substack{\Delta x\to 0\\ \Delta y=k_1\Delta x}}\left(\frac{u(x_0+\Delta\mathrm{x},\ y_0+\Delta\mathrm{y})-u(x_0,y_0+\Delta\mathrm{y})}{\Delta x}\right)\right) \tag{15}$$

Since the $\frac{u(x_0+\Delta\mathrm{x},\ y_0+\Delta\mathrm{y})-u(x_0,y_0+\Delta\mathrm{y})}{\Delta x}$ is a function only related to variable x changing, the partial derivative form is:

$$\frac{1}{1+ik_1}\frac{\partial u}{\partial x} \tag{16}$$

Since $\Delta y=k_1\Delta x$, denominator can be written as

$$\Delta z=\frac{\Delta y}{k_1}+i\Delta y \tag{17}$$

Then, take the $\frac{1}{i+\frac{1}{k_1}}$ off from formula (12):

$$\frac{1}{i+\frac{1}{k_1}}\left(\lim_{\substack{\Delta y\to 0\\ \Delta x=\frac{\Delta y}{k_1}}}\left(\frac{u(x_0,\ y_0+\Delta\mathrm{y})-u(x_0,y_0)}{\Delta y}\right)\right) \tag{18}$$

The partial derivative form is:

$$\frac{1}{i+\frac{1}{k_1}}\left(\frac{\partial u}{\partial y}\right) \tag{19}$$

Thus, the real part can be expressed as:

$$\frac{1}{1+ik_1}\frac{\partial u}{\partial x}+\frac{1}{i+\frac{1}{k_1}}\left(\frac{\partial u}{\partial y}\right) \tag{20}$$

The process of dealing with imaginary part is same as that of real part:

$$\mathrm{i}\left(\frac{1}{1+ik_1}\frac{\partial v}{\partial x}+\frac{1}{i+\frac{1}{k_1}}\left(\frac{\partial v}{\partial y}\right)\right) \tag{21}$$

Sorting out the formula:

$$\frac{dy}{dz} = \frac{1}{1+ik_1}\left(\frac{\partial u}{\partial x}\right) + \frac{k_1}{k_1 i+1}\left(\frac{\partial u}{\partial y}\right) + i\left(\frac{1}{1+ik_1}\frac{\partial v}{\partial x} + \frac{k_1}{k_1 i+1}\left(\frac{\partial v}{\partial y}\right)\right) \tag{22}$$

Since it is complicated to process if the coefficient is complex number, the numerator and denominator should time the conjugate of the denominator simultaneously.

$$\frac{dy}{dz} = \frac{1}{1+{k_1}^2}\left(\frac{\partial u}{\partial x}\right) + \frac{k_1}{1+{k_1}^2}\left(\frac{\partial u}{\partial y}\right) + \frac{k_1}{1+{k_1}^2}\left(\frac{\partial v}{\partial x}\right) + \frac{{k_1}^2}{1+{k_1}^2}\left(\frac{\partial v}{\partial y}\right)$$
$$+i\left(\frac{k_1}{1+{k_1}^2}\left(\frac{\partial u}{\partial x}\right) + \frac{-{k_1}^2}{1+{k_1}^2}\left(\frac{\partial u}{\partial y}\right) + \frac{1}{1+{k_1}^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{1+{k_1}^2}\left(\frac{\partial v}{\partial y}\right)\right) \tag{23}$$

The formula above is the derivative of complex function if it took its path from

If the path is from $y = k_2 x$, the derivative is same because the only difference is the slope of linear equation:

$$\frac{dy}{dz} = \frac{1}{1+{k_2}^2}\left(\frac{\partial u}{\partial x}\right) + \frac{k_2}{1+{k_2}^2}\left(\frac{\partial u}{\partial y}\right) + \frac{k_2}{1+{k_2}^2}\left(\frac{\partial v}{\partial x}\right) + \frac{{k_2}^2}{1+{k_2}^2}\left(\frac{\partial v}{\partial y}\right) + i\left(\frac{k_2}{1+{k_2}^2}\left(\frac{\partial u}{\partial x}\right) + \frac{-{k_2}^2}{1+{k_2}^2}\left(\frac{\partial u}{\partial y}\right) + \right.$$
$$\left.\frac{1}{1+{k_2}^2}\left(\frac{\partial v}{\partial x}\right) + \frac{1}{1+{k_2}^2}\left(\frac{\partial v}{\partial y}\right)\right) \tag{24}$$

If the derivative of complex function exists, equation (17) must be equal to equation (18) because limit has to converge in one point from every direction.

Using different symbols (a, b, c) to represent each coefficient of each partial derivative makes sorting the equation out much easier.

$$a = \frac{1}{1+{k_1}^2} \qquad b = \frac{k_1}{1+{k_1}^2} \qquad \mathrm{c} = \frac{{k_1}^2}{1+{k_1}^2} \tag{25}$$

For the limit from $y = k_2 x$, using (a_1, b_1, c_1) to represent each coefficient:

$$a_1 = \frac{1}{1+{k_2}^2} \qquad b_1 = \frac{{k_2}^2}{1+{k_2}^2} \qquad c_1 = \frac{{k_2}^2}{1+{k_2}^2} \tag{26}$$

$$a\frac{\partial u}{\partial x} + \mathrm{b}\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial x} + c\frac{\partial v}{\partial y} = a_1\frac{\partial u}{\partial x} + b_1\frac{\partial u}{\partial y} + b_1\frac{\partial v}{\partial x} + c_1\frac{\partial v}{\partial y} \tag{27}$$

Sorting this out:

$$b\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (a\text{-}a_1)\ \frac{\partial u}{\partial x} + (\mathrm{c}\text{-}c_1)\ \frac{\partial u}{\partial y} - b_1\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \tag{28}$$

Combining similar item$\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$:

$$(b - \mathrm{b}_1)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (a\text{-}a_1)\ \frac{\partial u}{\partial x} + (\mathrm{c}\text{-}\ c_1)\ \frac{\partial u}{\partial y} = 0 \tag{29}$$

For imaginary part:

$$-b\frac{\partial u}{\partial x} - c\frac{\partial u}{\partial y} + a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y} = -b_1\frac{\partial u}{\partial x} - c_1\frac{\partial u}{\partial y} + a_1\frac{\partial v}{\partial x} + b_1\frac{\partial v}{\partial y} \tag{30}$$

Sorting this out:

$$b\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + (c_1 - c)\frac{\partial u}{\partial y} + (a - a_1)\frac{\partial v}{\partial x} - b_1\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) = 0 \tag{31}$$

Combining similar item$\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)$:

$$(b - b_1)\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) + (c_1 - c)\frac{\partial u}{\partial y} + (a - a_1)\frac{\partial v}{\partial x} = 0 \tag{32}$$

Thus, after sorting this out, it is much easier to look for the relationship between the two equation (30) and (31).

$$c - c_1 = \frac{{k_1}^2}{1+{k_1}^2} - \frac{{k_2}^2}{1+{k_2}^2} = \frac{{k_1}^2-{k_2}^2}{(1+{k_1}^2)(1+{k_2}^2)} \tag{33}$$

And

$$a - a_1 = \frac{1}{1+{k_1}^2} - \frac{1}{1+{k_2}^2} = \frac{{k_2}^2-{k_1}^2}{(1+{k_1}^2)(1+{k_2}^2)} \tag{34}$$

Thus,

$$c - c_1 = a_1 - a \qquad \text{or} \qquad c - c_1 + a - a_1 = 0 \tag{35}$$

For real part:

$$(b - b_1)(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + (a - a_1)(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) = 0 \tag{36}$$

Since $b - b_1 \neq 0$ and $a - a_1 \neq 0$, which means that the coefficient of each term is none-zero, the formula has to satisfy the following equation:

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{cases} \tag{37}$$

For imaginary part, it uses same method used to deal with real part:

$$(b - b_1)(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}) + (a - a_1)(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = 0 \tag{38}$$

And the result also gets:

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{cases} \tag{39}$$

Based on the two identical results obtained by the equations of both real part and imaginary part, the C-R equation is always true regardless the limit paths. From the result of additional proof listed a above, it is as same as the result from C-R equation proved by tradition method (the one following $x - axis$ and $y - axis$)

2.5. The application of C-R equation

With the help of C-R equation, it is much easier to find out the number of differentiable points and their positions in complex plane based on the two conditions of C-R equation.

For example:

Consider the complex function:

$$f(z) = x^3 + i(2 - y)^3 \text{ where } z = x + iy \tag{40}$$

Thus,

$$u(x, y) = x^3 \quad \text{and} \quad v(x, y) = (2 - y)^3 \tag{41}$$

The partial derivatives of $u(x, y)$ with respect to x and y are:

$$\frac{\partial u}{\partial x} = 3x^2 \text{ and } \frac{\partial u}{\partial y} = 0 \tag{42}$$

The partial derivatives of $v(x, y)$ with respect to x and y are:

$$\frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = -3(2 - y)^2 \tag{43}$$

The two conditions of C-R equation are:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \tag{44}$$

The second condition in formula (41) is always true for every point because 0 is always equal to 0.

If the condition of C-R equation is satisfied, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$must be workable.

Thus,

$$3x^2 + 3(2-y)^2 = 0 \tag{45}$$

Only when $x = 0\ and\ y = 2$, the C-R equation is satisfied (because (2-y) cannot be negative number), which means that the only differentiable point in complex plane is when $z = 2i$. By C-R equation, it is easy to discovery that differentiability of $f(z) = x^3 + i(2-y)^3$.

In addition, the C-R equation can be used to determine the analyticity of complex function. If a complex function is analyticity, it has to both satisfy C-R equation and must be continuable everywhere. However, since $f(z) = x^3 + i(2-y)^3$ is differentiable only in one point, $f(z)$ is not analytic everywhere.

3. Conclusion

The paper showed the concept of limit of two variables function and the graphic meaning of partial derivative. In addition, the paper also introduced the four-dimensional complex function with real part and imaginary part. Based on the information discussed above, it is true that the C-R equation could be proved by traditional method, which makes the limit approach the x-axis and y-axis. However, its proof wasn't so rigorous because conventional method only considered two special cases. Thus, this paper gave an additional proof of the C-R equation by taking routes from $y = k_1 x$ and $y = k_2 x$.The C-R equation was strong in application in analytic function because this equation could easily the number of differentiable points and their positions in complex planes. Although this paper provides more rigorous, additional proof of C-R equation by following the route $y = kx$, this method has certain limitations. To be more specific, it cannot prove the feasibility of C-R equation by curved routes (i.e $y = x^2$). Thus, it is ongoing research.

References

[1] Qin Deng. Similarities and differences between complex function and real function [J]. Mathematics Learning and Research,2016(23):19.

[2] Qian Zhang. Discussion on Properties of complex Functions and bivariate Real Functions [J]. Journal of Jiamusi Education Institute,2011(01):159.

[3] Ma Guiying, Yang Shang. Compared with complex function of the real veriables function [J]. Journal of yinshan (natural science), 2011, 25 (3) : 97-98. The DOI: 10.13388 / j.carol carroll nki ysajs. 2011.03.023.

[4] Li Yuewu, Zhou Ruihong. Introduction of cauchy-riemann equation from a historical perspective [J]. Research of advanced mathematics,2009,12(04):121-123.

[5] Hong Xiao. A brief analysis of the concept of derivative in higher mathematics [J]. Mathematics Learning and Research,2021(32):8-10.

[6] Yu Zhai . Comparison of Properties of functions of complex variables and real variables [J]. Science and Technology Information,2013(12):109.

[7] LIU Yingwei. Supplementary proof of Cauchy-Riemann equation [J]. Education and Teaching Forum,2016(10):196-197.

[8] Qian Zhang Relationship between bivariate real function and complex function [J]. Journal of longdong university,2011,22(02):7-9.

[9] Xihua Gao. Introduction to the differentiability of the complex variable function and analyticity [J]. Journal of chifeng institute (natural science edition), 2013, 29 (16) : 8-10. DOI: 10.13398 / j.carol carroll nki issn1673-260 - x. 2013.16.112.

[10] Zou Quanchun. Dual function limit method [J]. Journal of reading and writing (education teaching publication), 2019 (10) : 21. DOI: 10.16071 / j.carol carroll nki/g4.2019.10.017 cn51-1650.

[11] Zhihui Zhang. Understanding of limit concept in Higher Mathematics [J]. Popular Investment Guide,2019(07):279.

[12] he yong. Geometric characteristics of directional derivatives of bivariate functions [J]. Journal of mathematical medicine and pharmacy,2014,27(03):270-271.

[13] Wang Peiying, Zhuo Haishan, Huang Simin. The continuity of binary function partial derivative calculation and visualization [J]. Journal of hengyang normal university, 2021, and (3) : 29-32. DOI: 10.13914 / j.carol carroll nki cn43-1453 / z. 2021.03.005.

[14] Zhong Yuquan, 2021. theory of functions of a complex variable. Beijing: higher education press. 274pp

[15] Wang Ridong, Tang Huo, Li Shuhai, Ao En, Zhou Haiyan, Liu Chunhui. Applied talents training mode of complex function visual teaching implementation [J]. Journal of chifeng institute (natural science edition), 2017 (19) : 4-7. DOI: 10.13398 / j.carol carroll nki issn1673-260 - x. 2017.19.002.

[16] Gao Xihua. Introduction to the differentiability of the complex variable function and analyticity [J]. Journal of chifeng institute (natural science edition), 2013, 29 (16) : 8-10. DOI: 10.13398 / j.carol carroll nki issn1673-260 - x. 2013.16.112.

[17] HUANG Yu-min. On Cauchy-Riemann Equation [J]. Journal of Zerling Teachers College,1981(01):85-90.