

Complex number and its discovery history

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Abstract. The operations of complex numbers are the main subject of the mathematical analysis area known as complex analysis. It is also known as the theory of functions of a complex variable. The primary research topic in the field of complex analysis is holomorphic functions. These functions are defined on the complex plane, have differentiable properties, and allow for negative values. The residue theorem, the Cauchy integral formula, the Laurent series expansion, etc. are a few concepts, ideas, and methods that are commonly used in research. In particular, over the years, complex analysis in mathematics, physics, and engineering has been extensively used in algebraic geometry, fluid dynamics, quantum mechanics, and other related areas. Two Italian mathematicians, Girolamo Cardano and Raphael Bombelli, discovered complex numbers in the 16th century while attempting to solve a particular algebra, and Cauchy and Riemann extended it in the 19th century. This essay begin with investigation of arithmetic property of complex numbers and then fully explores history development of complex numbers.

Keywords: Complex Plane, Cubic equation, Complex numbers.

1. Introduction

The set of complex numbers $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$ where $i = \sqrt{-1}$ ($i^2 = -1$). The coordinate plane \mathbb{R}^2 can be considered as the visualized form of \mathbb{C} where coordinate pair (x, y) plotted as $z = x + iy$. On this occasion, real axis and the imaginary axis are x-axis and y-axis. This coordinate plane is complex plane. According to the distance formula, in the complex plane, $|z|$ can be considered as the distance from z to O . Moreover, the distance from z to w is $|z - w|$. The addition can be visualized in complex plane as vector space addition on \mathbb{R}^2 . Besides, the way to visualize multiplication is more complex. Because the polar coordinates can represent every point in \mathbb{R}^2 . For any $z \in \mathbb{C}$ ($z \neq 0$). $\left|\frac{z}{|z|}\right| = |z| \cdot \left|\frac{1}{|z|}\right| = |z| \cdot \frac{1}{|z|} = 1$. As a result, $\theta \in \mathbb{R}$ exists and $\frac{z}{|z|} = \cos \theta + i \sin \theta$. As write $e^{i\theta} = \cos \theta + i \sin \theta$. Name of the equation $z = |z|e^{i\theta}$ is the polar form of z for $z \in \mathbb{C}$. The argument of z is the name of the angle $\theta \in \mathbb{R}$, which is denoted $\arg z$. Trigonometric identities can be used to check that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. Therefore, for $z \in \mathbb{C}$ $zw = |z|e^{i\arg z} \cdot |w|e^{i\arg w} = |zw| \cdot e^{i(\arg z + \arg w)}$. Thus, $|w|$ scales z and $\arg w$ rotates counterclockwise compose $z \rightarrow zw$. Moreover, counterclockwise rotation by $\arg(i) = \frac{\pi}{2}$ is corresponds to $z \rightarrow iz$.

Eventually, the real axis reflection is corresponded to complex conjugation $\overline{x + iy} = x - iy$. Since distance can be obtained by absolute value in complex planes, triangular inequality can be noted as $|a + b| \leq |a| + |b|$. From the perspective of inverse triangular inequality, it could be written as $||a| - |b|| \leq |a - b|$ and it can also lead to that $|a| = |a - b + b| \leq |a - b| + |b|$ From these inequalities, they will result in the following $|a| - |b| \leq |a - b|$ $|b| = |b - a + a| \leq |b - a| + |a|$ $|a| - (|a| - |b|) = |b| - |a| \leq |b - a| = |a - b|$. Most importantly, it get $|Im(a)| \leq |a|$ and $|Re(a)| \leq |a|$. Convergence is generally defined as a sequence $(W_n)_{new} \subset \mathbb{C}$ converges to $Z \in \mathbb{C}$ if $\lim_{n \rightarrow \infty} (W_n - Z) = 0$ which can be noted as $\lim_{n \rightarrow \infty} W_n = Z$ where Z was known as the limit of the sequence. Briefly, it can be written as $(W_n)_{new}$ converges to Z for $\forall n \geq n, |W_n - Z| < \epsilon$.

As $|Im(a)| \leq |a|$, $|Re(a)| \leq |a|$ and $|a| = (Re(a)^2 + Im(a)^2)^{1/2}$. For, $\lim_{n \rightarrow \infty} W_n = Z \rightarrow \lim_{n \rightarrow \infty} Re(W_n) = \lim_{n \rightarrow \infty} Im(W_n) = Z$. So, the notion of convergence agrees with the virtual one on R^2 .

$W_{nCN} \subset \epsilon$, a Cauchy sequence when satisfies $\forall \epsilon > 0$ and $\exists N \in N$, gives that $|W_n - W_m| < \epsilon$ in $\forall_{n,m} \geq N$. Thus, terms in Cauchy sequence can be conclude such that they would end up like what one person would expect them to be. For $P_0 \in \epsilon$ and $r > 0$, where r is the radius of an open disc centered at C_0 , the disc can be noted as $Pr(P_0) := \{P \in C: |P - P_0| < r\}$. Thus, such a closed disc can be written as: $\overline{Pr}(P_0) := \{P \in C: |P - P_0| \leq r\}$. And the circle with radius r would be: $C_r(P_0) := \{P \in C: |P - P_0| = r\}$. When any of the above is not important to the discussion, people can design the radius r or the center P_0 . Then it enables us to only keep the following part as the notation for the unit disk: $D := D_1(0) = \{P \in C: |P| < 1\}$. Next, the focus turns to the topological properties of C . Since the complex plane can be identified by R^2 and has the same concept of distance and convergence, these properties are just a transformation from R^2 to C . Most importantly, as can be seen from the examples above, this is not a dichotomy between open and closed: the whole is open or closed, or both. Usually, you can tell whether a set is open or not directly from the definition. Here's a handy way to see if the offer is over. Convergence of a sequence of complex numbers is defined in the same way as it is for real numbers. The necessary and sufficient condition for the sequence $\{Z_n\}$ to converge to ω is that the real and imaginary parts converge to the real and imaginary parts of ω , respectively. For all Cauchy sequences $\{Z_n\}$, when $n, m \rightarrow +\infty$, it holds that $|Z_n - Z_m| \rightarrow 0$. The set of real numbers is complete, the Cauchy sequence of real numbers converges, and the convergence in the range of complex numbers is equivalent to the convergence of two Cauchy sequence of real numbers. Thus the set of complex numbers is complete.

2. History of Complex numbers

The question of whether complex numbers exist is a tricky one. About three hundred years ago, the answer to this question was finally obtained. In fact, these numbers are quantities that arise in abundance in the study of mathematical disciplines, even though they may have meaning in reality or not at all.

Mathematics flourished gradually over thousands of years, but few people considered the limitations that only positive numbers and the real numbers imposed on mathematics and computation. In sixteenth century, imaginary number concepts were introduced. In the beginning, mathematicians did not even have the concept of negative numbers, let alone imaginary numbers. This led to the fact that even though they sometimes came up with similar ideas about complex numbers, they could not really touch on this concept. Nevertheless, certain "impossible" quantities were noticed by mathematicians working on cubic equations solutions. However, this almost completely foreign concept led them to believe that they had discovered something else. It is easy to see that the problem of "language barriers" permeates the work of mathematicians. As a result, it's not rare for them to clash with each other over the years. There is no standard by which to judge how these mathematicians think, and this makes each mathematician start almost from scratch. These factors have led to slow progress on some complex theories.

Over four centuries, whatever the original purpose, with increasing and intensive research, this complex system has been widely accepted as a mathematical truth. The questioning and testing of mathematics by mathematicians has led to the further development of its proofs. The emergence of more abstract proofs has given mathematicians confidence and continued the development of complex systems. Actually, this system is called "complex analysis". The development of complex numbers as an extension of rigorous proofs has provided new and expanding perspectives in many areas of mathematics [1-19].

The first traces of imaginary numbers were found in Italy, hidden in the cubic equation. In about 1530s, Nicolo Tartaglia, the mathematical genius, came on the scene [1-19]. Tartaglia often called a "stammerer" because of his speech impediment and the massacre that took place in his hometown as

a child. It ought to go without saying that Tartaglia was not taken seriously and his broad mathematical ideas were ignored.

The use of method for cubic equations of a certain form was one of his primary achievements [1-19]. His technique in his era was revolutionary. Tartar created multiple equations to express those ideas. Firstly, check the depressed cubic equation given by

$$x^3 + cx = d \quad (1)$$

Targaglia then defines two numbers whose difference is equal to d , and $(\frac{c}{3})^3$ is their product. Let's consider them as u and v . They become $u - v = d$; $uv = (\frac{c}{3})^3$.

The goal is still to find two numbers according to their product and difference. Tartaglia tried find the number $u + v$, and plus $u - v$ to solve variables. The method he used was to square the difference of u and v , add a quadruple product and then extract the square root [2]. Tartaglia found the solution creatively $x = \sqrt[3]{u} - \sqrt[3]{v}$. To apply Tartaglia's method on a numerical equation. Begin the equation like Equation 1, such as $x^3 + 3x = 4$. In order to find x , you have to solve u and v , they have the difference 4 and product 1. According to Tartaglia's method, solve $u + v$ firstly follow the step above. $u + v = 4\sqrt{(u - v)^2 + 4uv} = \sqrt{16 + 4} = \sqrt{20} \Rightarrow 2\sqrt{5}$. As $u + v = 2\sqrt{5}$ and $u - v = 4$, u can be solved by those two equations by setting them equally using substitution, then find v with the same process. Therefore, the solution of this cubic function can be found $x = \sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2}$. In 1539, Tartaglia's works were secretly passed on to Girolamo Cardano. However, Cardano published the method in his magnum opus *Ars Magna* [3]. As a result, after the publication of *Ars Magna*, the memory of Tartaglia's method disappeared, and the technique has since been known forever as Cardano's formula for depressed cubic. Cirolamo Cardano is considered as the most disgraceful mathematician. "Scoundrel" became synonymous with him and not without reason. In his magnum opus *Ars Magna*, the formula he developed for solving the depressed cubic using the Tartaglia's method. Cardano nearly insulted the Italian mathematician Ferro when his nook published. Nevertheless, the true creator Tartaglia was not praised [4]. Cardano's technique is expressed in special form. He showed his method of finding z in terms of a real cube. The representation of his idea is shown in three-dimensional. Cardano's approach is in the form of "rule" which is more generalized than using equations and steps. In his rule, he defined $x = t - u$ which is related to his cube, though Tartaglia's solution for z is still used. After Cardano geometrically solving his cube, he came up with a general solution [1-19].

Roots calculated by Cardano's method are shown in radical form, which it might cause the missing of some values. For the equation, $x = \sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2}$ mentioned formerly, it could be simplified to just $x=1$, however, by using Cardano's method, it would not be shown. Cardano's method is somehow not foolproof. In some cases, like what is in Equation 1, if containing some negativities, it would present results with radicand being negative without validating them. Since the non-real value are computed by the 'rules of ordinary arithmetic', scientists have been trying to prove their existence for more than three hundred years until they finally realized what they are proving is only imaginary quantity [1-19]. At that point, Cardano is not only trying to solve cubic equations in Tartaglia's version but also in a new form of parameterizing as $ax^3 + bx^2 + cx + d = 0$. However, by using his own method, Cardano continuously got roots that contains negative radicands, that he remained to assume no solution existing, while also missed the real roots that satisfies. His failure in solving cubic equations with imaginary quantities did not ruin his reputation and has also helped to sketch the blueprint of mathematical proof in the mid-sixteenth century. Cardano based his work on the structure of demonstration, rule, demonstration, rule, and so on (*Ars Magna*), and has helped to turn mathematics at that time from faith to prof. Using concrete proving, imaginary quantities could then be introduced. Rafael Bombelli, the 'last great sixteenth century Bolognese mathematician', also an engineer and architect, with his *L'Algebra*, known as the turning point of the development in complex number, started the rigorous proving in imaginary quantities. Bombelli found that Cardano's work useful but not necessarily correct as it is in 'real'. As the founder of complexity, he decided to

make up symbols himself [1-19]. He defined $\sqrt{-1}$ as *pdm*, in short of *piu di meno* in Italian, which means 'plus of minus' in English. Similarly, $-\sqrt{-1}$ was defined as *mdm*, *meno di meno*: 'minus of minus'. At that period of time, things in mathematics are still describe in words instead of symbols as symbols were not fully circulated. In his language of mathematics, 'plus of minus' times 'plus of minus' equals to 'minus' where 'minus' is negative one. After that, Bombelli starts to figure out cubic functions, especially those which can lead to roots of complex numbers. His procedure, accepting negative numbers as radicands which he named them as 'linked radicals of a new type', has well started the field of imaginary quantity as well as complex number. Through his own process of doing the irreducible cases of the cubic, he has shown the restriction of using Cardano's method of finding roots [1-19]. By showing that the combination of complex numbers can lead to real quantities, he has also provided the missing steps in Cardano's method. Bombelli, after this series of things, has started his own study in mathematics which was basically continuing to prove that by Cardano's recipe, real quantities can be manipulated, and that is what was known as conjugated complex number at present. In his belief, roots of cubic can be presented in form of $a + b\sqrt{-1}$; $a - b\sqrt{-1}$ where a and b are both real quantities. $2 + \sqrt{-121} = (a + b\sqrt{-1})^3 = a(a^2 - 3b^3) + b(3a^2 - b^2)\sqrt{-1}$. Bombelli through both sides cubic equation can be solved by a and b , Since the integer values of A and b need to be considered, and a must be either 1 or 2, meanwhile b must be either 1 or 11, because both 2 and 11 are prime numbers. The only two combinations of solutions that satisfy these two equalities are $a = 2$, $b = 1$. Find the solutions for a and b by substituting them into the original formula.

$x = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4$ and $x = 4$ just fits the previous inference of the original formula. This is why Bombelli was so great when he combined his theory with those of his predecessors to prove that real numbers are complex numbers. In fact, he demonstrated that any real number can be expressed as a complex number. This view was confirmed and accepted because he had the audacity to demonstrate a more specific proof idea. Instead of making rules out of calculations, he actually took something and showed why there was a misunderstanding about how the solutions of equations could be expressed differently. Thus, his introduction of complex numbers given mathematicians with a chance to extend cubic functions and to study algebra from a new perspective. By the end of the 17th century, Descartes was working on his symbolic rules, and also proposing the term "imaginable" for the complex, but without any positive discussion of the concept. In the 18th century, the concept of complex numbers came closer to our modern mathematical definition of them, they were more widely studied and adopted by popular mathematics, and during this century mathematicians had done enough to gain some confidence. The German mathematician Leibniz was one of the pioneers who took a step forward in the combination of numbers and shapes, especially in the geometric meaning of the conjugate of complex numbers. Inspired by Bombelli, Leibniz proposed that these cubic polynomials have either three real roots or two imaginary roots and one real root when studying the solutions of cubic polynomials. However, Leibniz's real contribution lies in the following The word "imaginable". Unlike Bombelli, who coined the word out of necessity, Leibniz can be explicit The reason why these quantities are called imaginary numbers. Although he used these numbers, he could not confirm their nature value. This goes back to early mathematics built on geometry. So, since there was little support for the geometric meaning of complex numbers, he said they weren't really numbers at all. It was from this perspective that the term "Imaginary" emerged and expanded its use. What makes Euler perhaps one of the greatest mathematical geniuses is that his ideas can be found in almost every field of mathematics today. He may not be known for his research on complex numbers, but he provided a basic but fundamental term for complex systems. He set $i = \sqrt{-1}$, and suggested that the best evidence that the problem could not be solved was to use this quantity to solve it. His creation of the word "i" did, however, make it simpler, but that did not prevent him from being one of the many mathematicians who still believed that complex numbers did not exist. As mathematicians began to study complex numbers in the 18th century, many reasonably came to believe that complex numbers had different "orders" or "types", because there were many different classes of real numbers such as rational and irrational, natural and integer numbers. However, D

'Alembert showed that there were many kinds of complex numbers, but they could all be expressed as $a+bi$, and this was the first time that a complex number was represented as two separate parts. This is also the contradiction of the complex number, which is infinite like the real number, but each of its elements can only be expressed in this single way [1-19].

3. Summary

Complex analysis is a branch of mathematical analysis that focuses on operations on complex numbers. The theory of functions of a complex variable is another name for it. Holomorphic functions are the main area of study in the subject of complex analysis. These functions may take negative values, have differentiable features, and are defined on the complex plane. The Cauchy integral formula, the Laurent series expansion, the residue theorem, and other ideas and techniques are some that are frequently employed in research. Algebraic geometry, fluid dynamics, quantum mechanics, and other related fields have employed complex analysis extensively over the years in mathematics, physics, and engineering. In the 16th century, two Italian mathematicians named Girolamo Cardano and Raphael Bombelli made the discovery of complex numbers while seeking to solve a specific algebra. In the 19th century, Cauchy and Riemann developed their work. This article begins by looking into the arithmetic properties of complex numbers and then make a thorough investigation of historical development of complex numbers. In the future, we will report more on topological properties of complex field and elegant properties of holomorphic functions.

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