Methods to Solve Laurent Series of Complex Functions

Chenhao Song*

Beijing Royal School, Beijing, 101121, China

*Corresponding author: songchenhao@st.brs.edu.cn

Abstract. In practical applications, there are always a lot of complex functions, which often make physicists and mathematicians very headache. It is a very important idea to use a polynomial function to approximate a complex function. This paper is about the Laurent series of complex functions, which includes holomorphic functions and meromorphic functions, and tries to derive a general formula or a general way to compute the Laurent series. This paper also includes several basic examples of different types of complex functions and the process to derive the Laurent series of these functions. By finding similarities and summarizing the process, a general method for each or overall function is shown. Examples include common fractional functions, Trigonometric functions, and exponential functions. Some of them include different ways of computing according to it's characteristics. An analytic function can be extended to a function obtained by analytic extension of the Taylor series defined on an open region on the complex plane, and this method of complex analysis is feasible.

Keywords: Laurent series, Taylor series, Analytic function.

1. Introduction

In the complex functions, some new definitions are not in normal real functions, such as modulus, argument, and the circular range of the function [1]. Complex functions also have the similar process of computing derivatives and integrals, but there are some differences that make people use new methods to do with it [2].

Laurent series is an important and basic knowledge related to complex functions [3]. It shows the function in a term of the sum of infinite terms. In a closed ring range, the Laurent series is very important [4-6]. By using the Laurent series, residue can be easily found and used to compute integrals of the closed area [7]. In the negative power part of the Laurent series, the -1 term represents the residue of the function and that is used to calculate integrals of the closed area by time residue to [8]. In some of the calculations, using residue to calculate integral is a much easier way than calculating directly [9].

Even though the connection with complex functions is cleared, it is not easy to compute the Laurent series of a complex function [10]. Holomorphic functions are the same as analytic functions, and the rest of the content may main about analytic functions. The author tries to find out a general way to solve the function in Laurent series form.

2. Basic Concepts

2.1. Taylor Series and Laurent Series

The Taylor series is the function that shows the sum of infinite terms expressed by the derivatives at a single point of a function. It is similar to the Laurent series which does not have negative power terms. The general equation of the Taylor series is

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{1}$$

Laurent series represents functions in a form for the sum of infinite terms of derivatives in the function. It contains infinite positive power terms and infinite negative power terms. It can also be used to find the residue and the integral of the complex function. Usually, it is expanding in a ring region but sometimes there are exceptions. Below is the general equation of the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, where the coefficient $a_n = \frac{1}{2\pi i}\oint_c f(z)\frac{1}{(z-z_0)^{n+1}}$. As mentioned above, the general equation of the Laurent series actually contains the positive power term part and the negative power part, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \ + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}, (R_1 < |z-z_0| < R_2). \tag{2}$$

2.2. Analytic Functions/Holomorphic Functions

When a complex function is “differential”, it is called analytic. Some functions have points or part which is not analytic, the part is called the singularity. When a function is analytic in all parts of it's domain, or the function does not have any singularity, it is called an analytic function or holomorphic function.

3. Methods to Obtain Laurent Series

3.1. Fractional Equation

Taylor series and Laurent series have similarities. When computing Laurent series, some of the common Taylor series may be used in order to simpler compute Laurent series. There is an equation of Taylor series which can be used to solve the function. It is useful because of lots of fractional equations, such kinds of functions can simply solve by changing their denominator to the form that is similar to it. In addition, in this kind equations, in other words, the term of at the same place in this equation should has a modulus less than 1, no matter what it is. For example, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. As a first example, this article considers the function

$$f(z) = \frac{1}{(z-4)(z-9)}. \tag{3}$$

The domians of this question are:(1) $0 < |z| < 4$, (2) $4 < |z| < 9$ and (3) $9 < |z| < +\infty$. The first several steps are some for those 3 domains. First, it can be separated into two fraction$= -\frac{1}{5}\frac{1}{z-4} + \frac{1}{5}\frac{1}{z-9}$. Then this article times -1 for the second part and combine the function$= \frac{1}{5}\frac{1}{4-z} - \frac{1}{5}\frac{1}{9-z}$. The thing is, the denominator of the processed function is not as same as the Taylor series, and it must turn to a form of 1-z (z can be other forms). The next step is going to change the form of the denominator so it can be processed easily. Next, this article separates the function into two parts and process them separately. But when changing the form of the denominator, it should be fixed into the domains that are given. In both parts of the functions, the coefficient will not be considered in the first time. It will be timed into the function in the last step, in order to simplify the computing process.

Firstly, this article considers the first domains $0 < |z| < 4$. In the front part of the function, $\frac{1}{5}$ will not be considered in the first time, and the rest parts is divided by 4 on its molecular and denominator $\frac{\frac{1}{4}}{\frac{4-z}{4}} = \frac{1}{4}\times\frac{1}{1-\frac{z}{4}}$. In the denominator, the domian needs to be included. In the begining, it already stated that $\left|\frac{z}{4}\right| < 1\ so\ |z| < 4$. Even though it is not fully contained by $0 < |z| < 4$, but it contains $0 < |z| < 4$, so this answer is still acceptable. The second part of the function undergoes the same process$\frac{\frac{1}{9}}{\frac{9-z}{9}} = \frac{1}{9}\times\frac{1}{1-\frac{z}{9}} = \frac{1}{9}\sum_{n=0}^{\infty}(\frac{z}{9})^n$ $\left|\frac{z}{9}\right| < 1, so\ |z| < 9$. Both equations are acceptable in the domain $0 < |z| < 4$, because $|z| < 9$ comtains the required domain so it is the correct way.

Then in the first part, this article can replaces $\frac{z}{3}$ in to the place z in the Taylor series equation, then the answer should be $\frac{1}{4}\sum_{n=0}^{\infty}(\frac{z}{4})^n$. To combine the two parts of the equation together and time the initial coefficient of each part of function, the totally answer should be:

$$\frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n-\frac{1}{45}\sum_{n=0}^{\infty}\left(\frac{z}{9}\right)^n 0<|z|<4. \tag{4}$$

Seconly, this article focuses on the second domain $4<|z|<9$. The former steps are same, so it can be started at here $f(z)=\frac{1}{(z-4)(z-9)}=\frac{1}{5}\frac{1}{4-z}-\frac{1}{5}\frac{1}{9-z}$, the coefficient will not be considered in the first time. For the first part of the function, it should be firstly tested by the first approach. $\frac{1}{4-z}=\frac{1}{4}\times\frac{1}{1-\frac{z}{4}}$ $\left|\frac{z}{4}\right|<1\ so\ |z|<4$. Obviously, it is not in the domain of $4<|z|<9$ any more, so it needs another way to process the function in order to fit into the domain. In this case, z of 4-z of the denomitor can change to 1 by deviding z on denominator and molecular, to make the denominator to meet the condition. $\frac{1}{4-z}=-\frac{1}{z}\times\frac{1}{1-\frac{4}{z}}$ $\left|\frac{4}{z}\right|<1\ so\ |z|>4$. Then it fits into the domain of $4<|z|<9$. Then, this article tests the second part of the function by using the first approach. $\frac{1}{9-z}=\frac{1}{9}\times\frac{1}{1-\frac{z}{9}}$ $\left|\frac{z}{9}\right|<1, so\ |z|<9$. It is still fit into the domain of $4<|z|<9,$ so it does not need to change. In this case, it is the correct answer because it is in the domain, so the overall answer should be

$$-\frac{1}{5z}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n-\frac{1}{45}\sum_{n=0}^{\infty}\left(\frac{z}{9}\right)^n 4<|z|<9 \tag{5}$$

Finally, this article studies for the third domain $9<|z|<+\infty$. It can also start with the middle step $f(z)=\frac{1}{(z-4)(z-9)}=\frac{1}{5}\frac{1}{4-z}-\frac{1}{5}\frac{1}{9-z}$. Still, the first part of the function should be test with the approach used in the former question. $\frac{1}{4-z}=-\frac{1}{z}\times\frac{1}{1-\frac{4}{z}}$ $\left|\frac{4}{z}\right|<1\ so\ |z|>4$. Then this article turns to test the second part by using the same way used in the former question. $\frac{1}{9-z}=\frac{1}{9}\times\frac{1}{1-\frac{z}{9}}$ $\left|\frac{z}{9}\right|<1, so\ |z|<9$. However, in this case, this method is not acceptable anymore because it is out of domain, so the method should be changed. $\frac{1}{9-z}=-\frac{1}{z}\times\frac{1}{1-\frac{9}{z}}$ $\left|\frac{9}{z}\right|<1\ so\ |z|>9$. Now this function is fitted into the domain of $9<|z|<+\infty$. Therefore, the overall answer should be

$$-\frac{1}{5z}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n-\frac{1}{5z}\sum_{n=0}^{\infty}\left(\frac{9}{z}\right)^n 9<|z|<+\infty \tag{6}$$

On the other hand, there is another example where

$$f(z)=\frac{z}{z^2(z^2-1)}(1)0<|z|<1(2)1<|z|<+\infty. \tag{7}$$

Sometimes the numerator is also included when processing the function. In this function, and some simliar functions, there are some ways to cancel out the numerator in order to simplify the function in the first time. In this function, obviously, the denominator can be rewritten into $z^2(z^2-1)=z^2(z+1)(z-1)$ and the numerator can also be changed in order to help for cancelling out by z+1-1. So totally this function can be seperated into two parts: $\frac{1}{z^2(z+1)}+\frac{1}{z^2(z^2-1)}$. Then keeping processing the function: $\frac{1}{z^2}(\frac{1}{z+1}-\frac{1}{2}\frac{1}{z+1}+\frac{1}{2}\frac{1}{z-1})$, the functions inside the quote can convert series form, by using the Taylor series of $\frac{1}{1-z}$. The modulus of the value at the place of z is less than 1 ($|z|<$

1). Following computing is focused on the function inside the quote and $\frac{1}{z^2}$ will be includeD in the computing in the last step.

Firstly, this article considers the first domain $0 < |z| < 1$. No matter the fraction is $\frac{1}{z+1}$ or $\frac{1}{z-1}$, they are no need to change to form of the z term in the denominator to fit in to the Taylor series of $\frac{1}{1-z}$. The only thing needed to do is to change the sign on the denominator to arrange the order. So firstly the domain of this function should be sure to fit it the domain required. In the Taylor series of $\frac{1}{1-z}$, $|z| < 1$, it is already be acceptable in the domain of $0 < |z| < 1$, so it is acceptable. The series form of the function inside the quote should be:

$$\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{2}\sum_{n=0}^{\infty} z^n \tag{8}$$

Seconly, this article focuses on the second domain $1 < |z| < +\infty$. In the second domain, the form of the denominator must be change, because $|z| < 1$ is not in the domain any more. So it should be processed in different way. Similar with the former question, the second domain. If $|z| > 1$, by deviding z on the denominator, for example, $\frac{1}{z+1}$ turns to $\frac{1}{1+\frac{1}{z}}$ while $\left|\frac{1}{z}\right| < 1$ is acceptable becasue in this case $|z| > 1$ and the it is fitting in the domain. And the other steps undergo the same process. The final answer should be:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^n} \tag{9}$$

Before ending this part, this article summarizes the way to process fractional functions. Firstly, the first kind of complex function, fractional function, is harder to compare with other kinds of functions because it is hard to separate the function into several shorter and easier fractions. Normally, it is easy to separate the denominator of the function into several factors, while it is hard to determine the numerator of each factor. Here is a general way of this process:

$$\frac{1}{(x-n)(x-m)} = \frac{A}{x-n} + \frac{B}{x-m}. \tag{10}$$

While using this method, the numerator is not included in thinking, but only replaced with A and B after the fraction is separated. Then conduct the fractions to a common denominator, so A and B are timed with the other fraction's denominator and the value of A and B can be known because A and B should be able to make their value correspond to the original numerator. When the function is factorization successfully, then $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \ |z| < 1$ is used to do the final calculation. The domain of the function is very important because the Laurent series is expanded in a ring range and some functions have more than one singularity while some do not have any. Obviously, in the first question of 3.1, the different domains caused completely different results. In order to change the form of the fraction to fit the domain, the main thing needed to take care of is that the "$|z|$" must be smaller than 1 (any term of z is acceptable, if only the same place of z in $\frac{1}{1-z}$), and fit into the domain.

3.2. Trigonometric Functions

There are some other kinds of functions that can be solved easily when compared to the Taylor series, such as trigonometric functions. It will be much more complex than the example just mentioned. There is an example of the trigonometric functions

$$\sin\left(\frac{z}{z-2}\right), 2 < |z| < \infty. \tag{11}$$

Firstly, it will be simple the fracion inside the quote of sine $\frac{z}{z-2} = \frac{z-2+2}{z-2} = 1 + \frac{2}{z-2}$. Then replace the new equation to the original equation $\sin(1 + \frac{2}{z-2})$. In order to seperate the equation and make it simpler, here is an equation which can speperate the equation into two parts which is: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$. So the equation can be seperated into two parts: $\sin(1)\cos\left(\frac{2}{z-2}\right) + \cos(1)\sin(\frac{2}{z-2})$. As $\sin(1)$ and $\cos(1)$ turn to be two constant value, it will not be include of computation for a while. Here are two equations of Taylor series of sine and cosine, which will be to compute

$$\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}, \cos(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}. \tag{12}$$

Then replace $\frac{1}{z-1}$ to the place of z in the upper equations

$$\sin(1)\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{(z-2)^{2n}(2n)!} + \cos(1)\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(z-2)^{2n+1}(2n+1)!} \tag{13}$$

Before ending this part, this article summarizes the way to process trigonometric functions. The way to compute trigonometric function is also required to change the original function and allow it to be replaced by the Taylor series. The unique part of trigonometric functions is that there are some equations that represent the sum of angles, the difference of angles, the product of angles, and so on. Then in the same step, replace the Taylor series into the function and finally arrange the function to the Laurent series.

3.3. Exponential Functions

The symbol e is a constant value, and it is usually included in the functions and it also has unique characteristics. It also has the Taylor series which can be use to help computing Laurent series. The Taylor series related to it is $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. Here is an example:

$$\frac{e^{\frac{1}{z^2}}}{z^3}, 0 < |z| < +\infty. \tag{14}$$

In this question, in order to simplify the function, this function is very similar with the derivative of $e^{\frac{1}{z^2}}$ (the negative sign is missed when compared to the actual derivative of $e^{\frac{1}{z^2}}$). In this case, obviously if the function can turn to the term before the derivative, it will be much easier to process. So the function can be rewrite as $\frac{e^{\frac{1}{z^2}}}{z^3} = -\frac{d}{dz}e^{\frac{1}{z^2}}$. The Laurent series of $e^{\frac{1}{z^2}}$ is $\sum_{n=0}^{\infty}\frac{1}{z^{2n}n!}$, so the function should turns to be $\frac{e^{\frac{1}{z^2}}}{z^3} = -\frac{d}{dz}\sum_{n=0}^{\infty}\frac{1}{z^{2n}n!}$.

After the calculation, the final answer of the Laurent series of this function should be:

$$\sum_{n=0}^{\infty}\frac{1}{z^{2n+1}(n-1)!}. \tag{15}$$

Before ending this part, this article summarizes the way to process exponential functions. When processing exponential functions, especially the exponential functions related to (natural constant), sometimes it can be simplified by finding out that they are differential of another exponential function. The exponential functions usually contain part of its original function before taking derivative, so when the function is reversed to its original form, this equation $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ can be used.

3.4. Laurent Series of Holomorphic and Meromorphic Functions

The definition of holomorphic functions and meromorphic functions have already been explained. Laurent series contains positive and negative power terms because it represents both the analytic part of the function and the unanalytic of the functions. Sometimes the Laurent series does not have negative power terms, it is possible because that means the function is analytic. Here are two examples of the holomorphic function and meromorphic function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and

$$f(z) = \frac{1}{(z-4)(z-9)} = -\frac{1}{5z}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n - \frac{1}{45}\sum_{n=0}^{\infty}\left(\frac{z}{9}\right)^n, 4 < |z| < 9. \tag{16}$$

These two are the Taylor series and the example shown in the former contents. e^x is analytic in the plain, so there are negative power terms appeared. In other words, this function does not have singularities so it is analytic, indicating that it does not have negative power terms.

In the second example, an example question shown before, even though n start at 0 rather than $-\infty$, but in this part of Laurent series of the function $\sum_{n=0}^{\infty}(\frac{2}{z})^n$, when n increase, the power of z is negative and continuously increase negatively. In the domain of $4 < |z| < 9$, the function is expanding in a ring range, and does have sigularities, so it has negative power terms in this domain but not in the domain of $0 < |z| < 4$.

4. Summary

Generally, when computing Laurent series of complex functions, one should know its connection with the Taylor series as the latter is easy to process or already known. In order to simplify the function, Taylor series are always including in computing the complex function turn to Laurent series. From the upper content, it is easy to find out that the main things needed to do is to convert the original function to the form that can be easy to solve by using Taylor series. In addition, the way of simplifying the function is not limited in one, several ways are acceptable in algebra. When processing the final step, the series is related to the domain of the function. It needs to be followed the domain of $|z|$ in the original Taylor series. It should be carefully processed because the domain changes and the answer are not fit in all domains.

This essay is mainly focused on the simple equations that are easy to simplify with algebra-based knowledge or simple calculus. However, there are still other kinds of questions that are the combination of different kinds of the question shown in the former contest. When it happens, the combination of the methods should also be used during the computation.

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