Residue Theorem and Its Application

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Abstract. Sometimes it can be frustrating to get the anti-derivatives of some functions. Some definite integrals are difficult, or even impossible, to calculate using the elementary calculus method, especially in the field of complex analysis. In such cases, Cauchy's Residue Theorem might be proper. In this article, we state the difference between Cauchy's Integral Theorem and the Residue Theorem, followed by the definitions of isolated singular point and residue with some examples given after. These are prerequisites for understanding the theorem. A short statement of the content of Cauchy's Residue Theorem has been made, according to which the integral of a function along a positively oriented simple closed contour is related to the residues, which can be obtained through Laurent expansion, of all the limited isolated singular points inside the contour. In addition, using the Cauchy-Goursat theorem, we developed the derivation of CR Theorem. It denotes that the value of a function around a simple closed contour is zero if the function is analytic at all points interior to and on that contour. Then we launch an application and accomplish it in two ways. The first method makes use of the Fourier Matrix, while the second is related to power series. In both of these approaches, we feel the efficacy of Cauchy's Residue Theorem.

Keywords: Cauchy's Residue Theorem; Isolated Singular points; Residue.

1. Introduction

Cauchy’s Residue Theorem is a continuation of the Cauchy Integral Theorem. It’s closely related to the Taylor series and Laurent series, which is essential in analytic-function. An analysis of the CR Theorems reveals that the composition of the theorem is actually Cauchy's Integral Theorem in form under the condition of isolated singularities and the application of complex series in complex integration. However, in contrast to Cauchy's Integral theorem, which only gives an abstract circumferential integral, the CR Theorem directly states that the value of the enclosing integral is directly related to the singularity in the enclosed path, and can be calculated using Laurent expansion.

Cauchy-Goursat theorem shows that if all points of a function inside and on a simple closed contour Care analytic, then the value of the function surrounding the contour is zero. If the function fails to be analytic at a finite number of points inside contour C, that number is called the residue, and all of the points contributes to the value of the integral.

The Residue Theorem of Cauchy is developed. CR Theorem arises in this case when calculating some distributions that are impossible to calculate using elementary calculus. In the calculation, we represent this integral as the limit of a path integral using CR Theorem. At the same time, using the CR Theorem, a considerable number of definite integrals can be calculated, the integrals that cannot be calculated with basic method can be solved. The essential background of complex analysis can refer to [1-4], which illustrate some basic understanding of complex numbers and further explanations.

2. Isolated Singular points and Residue

The definition of isolated singular points is that if no other singular points exist within a neighborhood of radius of a singularity, this singular point is considered to be isolated. For example,

$$\mathrm{f}(z) = \frac{z-1}{z^5(z^2+9)} \tag{1}$$

it has three isolated points which is z=0 and z=$\pm$3i. M ore details can refer to [5].

IS points can be defined into three types, which are removable singular point, essential singular point and poles of order m. They are differentiated by the number of negative power terms of the Laurent expansion. The expansion of Laurent series can refer to [7-9]. The Laurent expansion has zero negative power terms for removable singular points. The Laurent expansion has an infinite number of negative power terms at the essential singular point. Finally, poles of order m have a finite number of Laurent expansion negative power terms. Zero and poles can also be used to identify the type of IS point.

For zero of order m,

$$\mathrm{f}(z) = \mathrm{g}(z)(z - z_0)^m,\ \ \mathrm{g}(z) \neq 0, \tag{2}$$

or

$$\mathrm{f}(z_0) = f'(z_0) = \cdots = f^{m-1}(z_0),\ \ f^m(z_0) \neq 0 \tag{3}$$

z_0 is the zero of order m of $\mathrm{f}(z)$.

For pole of order m,

$$\mathrm{f}(z) = \frac{g(z)}{(z-z_0)^m},\ \ \mathrm{g}(z) \neq 0, \tag{4}$$

When $\mathrm{f}(z) = \frac{f_1(z)}{f_2(z)}$, z_0 is the zero of order m of $f_1(z)$ and the zero of order m of $f_2(z)$, if $\mathrm{m} \geq n$, z_0 is a removable singular point of $\mathrm{f}(z)$.

Some common examples such as $e^{\frac{1}{z}}, \sin\frac{1}{z}\, and \cos\frac{1}{z}$ belong to essential singular point after considering the number of negative power terms of the Laurent expansion where it is the Taylor series of the function at x=0. With the method of identifying types of IS points, some examples are shown. For example, consider the type of IS points of the followings:

Example1 $$\mathrm{z} = 0, \mathrm{z}\left(e^{\frac{1}{z}} - 1\right) \tag{5}$$

Consider the Laurent expansion of this function is the first step, as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \mathrm{z} + \frac{z^2}{2!} + \ \ , \tag{6}$$

the expansion of $e^{\frac{1}{z}}$ is that

$$e^{\frac{1}{z}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} +, \tag{7}$$

so that $e^{\frac{1}{z}} - 1$=$z^{-1} + \frac{z-2}{2!} + \frac{z^{-3}}{3!} +$,z$\left(e^{\frac{1}{z}} - 1\right)$=$1 + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} +$. After counting the number of negative power terms of the Laurent expansion, it can be deduced from the conclusions above that z=0 is an essential singular point because that infinite number of negative power terms of Laurent expansion are shown.

Example2 $$\mathrm{z} = 0,\ \ \frac{\sin z}{z} \tag{8}$$

Consider the Laurent expansion of this function is the first step,

$\mathrm{sinz} = \mathrm{z} - \frac{1}{3!}z^3 \frac{1}{+5!}z^5 +$, so that $\frac{\sin z}{z} = 1 - \frac{1}{3!}z^2 \frac{1}{+5!}z^4 +$. The number of negative power terms of the Laurent expansion is zero, so it is clear that z=0 is a removable point from previous conclusions.

Example3 $$z = 0,\ \ \mathrm{f}(z) = \frac{e^z-1}{z}, \tag{9}$$

So, $f_1(z) = e^{z-1}, f_2(z) = \mathrm{z}$, m=1, n=1. As m=n, z_0 will be a removable singular point of $\mathrm{f}(z)$.

The examples above illustrate how to identify the types of IS points using different methods, now it turns to the concept of residue.

Residue is a necessary concept in complex functions, the value is which the integral value of analytic function of any closed curve that encloses an isolated singularity divided by 2πi. Residue is often applied to certain special types of real integrals, thus greatly simplifying the calculation process of integrals. Depending on the isolated singularity, different calculation methods are used. The more rigorous explanation is that when z_0 is an IS point of a function f, there is a positive number r where f is analytic at each point z for which $0 < |z - z_0| < \mathrm{r}$ [1]. In this case, the Laurent expansion of $\mathrm{f}(z)$ is:

$$\mathrm{f}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z-z_0} + \frac{b2}{(z-z_0)^2} + \cdots + \frac{bn}{(z-z_0)^n} \tag{10}$$

The coefficients of a_n and b_n have their representations. For b_n,

$$b_n = \frac{1}{2\backslash pii} \int_c \frac{f(z)dz}{(z-z_0)^{-n+1}} \text{ (n=1,2, ...)} \tag{11}$$

C is possible oriented closed contours around z_0 which lies in $0 < |z - z_0| < \mathrm{r}$. When n=1, the expansion turns to

$$b_1 = \frac{1}{2\backslash pii} \int_c f(z)dz \tag{12}$$

or

$$\int_c f(z)dz = 2\pi\mathrm{i} b_1 \tag{13}$$

In this case, complex number b_1 is defined the residue of f at the isolated singular point z_0, which is the coefficient of $\frac{1}{z-z_0}$ in the previous expansion.

There are many ways to calculate the residue of IS points. The methods for calculating the residue for different isolated singularities also vary, but the most general approach is to calculate it by using the definition of the isolated singularity, which is also the first term of the main part of the Laurent expansion of the function. For example, consider the residue of the functions at z=0

Example1 1

$$\frac{1}{z+z^2} \tag{14}$$

It is obvious that that $\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z}$ and $\frac{1}{1+z}$ can be expanded, so according to the expansion, $\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z}(1 - z + z^2 - \cdots) = \frac{1}{z} - 1 + \mathrm{z} -$ the residue of z=0 is the coefficient of $\frac{1}{z}$, which is 1.

Example1 2

$$\mathrm{z}\cos\left(\frac{1}{z}\right) \tag{15}$$

The expansion of $\cos(z)$ is that $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} +$, so it can be deduced that $\cos\left(\frac{1}{z}\right) = 1 - \frac{z^{-2}}{2!} + \frac{z^{-4}}{4!} - \frac{z^{-6}}{6!} +$, when it times z, it is clear that that the expansion of $\mathrm{z}\cos\left(\frac{1}{z}\right)$ is $\mathrm{z}\cos\left(\frac{1}{z}\right) = \mathrm{z} - \frac{z^{-1}}{2!} + \frac{z^{-3}}{4!} - \frac{z^{-5}}{6!} +$, in this case, the residue of z=0 is $-\frac{1}{2}$, which is the coefficient of $\frac{1}{z}$.

Example1 3

$$\frac{z - \sin z}{z} \tag{16}$$

It is easy to get that $\frac{z-\sin z}{z} = \frac{1}{z}(z - \sin z)$, considering the expansion of sinz, it can be deduced that $\frac{1}{z}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right] = \frac{z^2}{3!} - \frac{z^4}{5!} +$, the residue at z=0 is 0 because that the coefficient of $\frac{1}{z}$ in the expansion is 0.

Examples above demonstrate how to solve the residue at the point z=0, considering the integral using the residue and the equations (4) and (5) would be a more complicated case which is presented in the next question.

For example, consider the integral

$$\int_c \cosh\left(\frac{1}{z^2}\right) dz \tag{17}$$

Cosh is a hyperbolic function which is a class of functions. The hyperbolic sinh and the hyperbolic cosh are the most basic hyperbolic functions, from which the hyperbolic tangent function tanh and the like can be derived, the derivation of which is also similar to the derivation of trigonometric functions.

In this case, C is a positively oriented unit circle $|z|$=1. Except the origin, the function is analytic in the plane, the IS point z = 0 will be interior to C. The value of the integral is $2\backslash pi$i times the residue at z=0.

According to Maclaurin series,

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, (|z| < \infty) \tag{18}$$

So that,

$$\cosh\left(\frac{1}{z}\right) = 1 + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{4!} \cdot \frac{1}{z^4} +, \ (0 < |z| < \infty) \tag{19}$$

The residue of its IS point =0 is 0, so that the result of 0 time $2\backslash pi$i is 0, the result of the integral equals zero due to the fact that the value of the integral equals $2\pi i b_1$

3. Cauchy's Residue Theorem

We can easily deduce that the singular points within a simple closed contour C must be isolated if a function f is analytic everywhere inside C except for these specific finite singular points based on the definitions of isolated singular points and residues. This deduction is one of the prerequisites for deriving the Cauchy's Residue Theorem.

Imagine this simple connected contour C on the Figure.1 below, described in positive sense. There are some points scattered inside the contour C and they are named Z_k (k=1, 2, 3, ..., n), which are also the centers of positively oriented circles C_k (k=1, 2, 3, ..., n). These circles are all set in the interior of C and they are small enough so that each two of them do not share common points. In other words, they are isolated. The larger simple closed contour C and the circles C_k work together to define the area where the function f is analytic. The region inside of the contour C and simultaneously outside of the circles C_k makes up the interior of this region, which is a multiply connected domain.

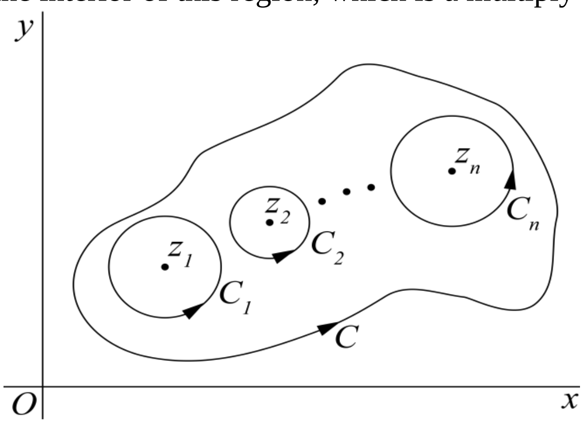

Fig.1 A Simple Closed Contour C

In the second section, we mentioned that the Cauchy-Goursat theorem states that when a function is analytic at all points inside of and on C, the value of the integral around a simple closed contour is zero. By applying Cauchy-Goursat theorem to this region, we can get

$$\int_C f(z)d\,z - \sum_{k=1}^{n} \int_{C_k} f(z)d\,z = 0 \tag{20}$$

Thus, we can get the fact which is precisely what Cauchy's Residue Theorem indicates: in a positively oriented simple closed contour C, the integral of f along C is 2πi times of the sum of the residues of all the limited isolated singular points inside the contour C, with k=1, 2, 3, ..., n. The theorem can be expressed as:

$$\int_C f(z)d\,z = 2\pi i \sum_{k=1}^{n} Res\, f(z)_{z=z_k} \tag{21}$$

4. Application

By using these theorems, the value of some more complicated contour integrals can be found(the definition of contour integral can be refer to ref[5]).

4.1. Example:

$$\int_{c(t)} \frac{f(z)}{z^4-1} \mathrm{dz} \tag{22}$$

In which,

$$f(z) = 10z^3 + (2i-2)z^2 + 2z + 2 - 2i \tag{23}$$

is a complex number, $\mathrm{c}(t) = 2e^{it}$ or $\mathrm{c}(t) = 1 + e^{it}$ or $\mathrm{c(t)} = \frac{1-\mathrm{i}}{2} + \sqrt{2}\,\mathrm{e}^{\mathrm{i\,t}}$, t is a real number. $(0 < \mathrm{t} \leq 2\pi)$

4.2. Method by using Fourier Matrix.

4.2.1 Fourier matrix

It is the matrix form of Fourier transform, which also is a kind of mathematical tool that can be applied to the case we are trying to work out. We can use it to reduce computational consumption and calculate A, B, C and D easier. A Fourier matrix F_n is seem like

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^1 & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix} \tag{24}$$

In which, w is belong to complex domain. And it has this property:

$$\text{if} \quad \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} F_n = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \text{ then } \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = F_n^{-1} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{25}$$

More detail about the Fourier matrix and Fourier transform can refer to [6].

Firstly, we factor $z^4 - 1$, such that:

$$\mathrm{z^4 - 1 = (z^2 - 1)\big(z^2 - (-1)\big) = (z+1)(z-1)(z+i)(z-i)} \tag{26}$$

Which mean $\frac{f(z)}{z^4-1}$is not differentiable (in this case, ‘differentiable’ means ‘analytic’, the same below.) in z=i, z=1, z=-i and z=-1, since the value of $\frac{f(z)}{z^4-1}$ does not exist when z equal to these values.

Meanwhile, $\frac{f(z)}{z^4-1}$ is also not differentiable in (1,0), (0, i), (-1,0), (0, -i) (these are the correspondence points of z=i, z=1, z=-i and z=-1in the complex plane).

4.2.2 Finding the singular points of $\frac{f(z)}{z^4-1}$ for each c(t)

As the introduction and the explanation of singular points and Cauchy's Residue Theorem (point.2 refer to ref[1]), it is necessary to find the singular points for each simple closed contour c(t)as trying to find the integral of $\frac{f(z)}{z^4-1}$ (the definition of simple closed contour is refer to ref [10]).

We can assume that

$$g(z) = \frac{f(z)}{z^4-1} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z-i} + \frac{D}{z+i} \quad (27)$$

4.2.2.1 The isolated singular point for $c(t) = 2e^{it}$

Observe the curve of $c(t) = 2e^{it}$ (which is showed in Figure.2), we notice that (1,0), (0, i), (-1,0), (0, -i) are all inside the curve of $c(t) = 2e^{it}$.

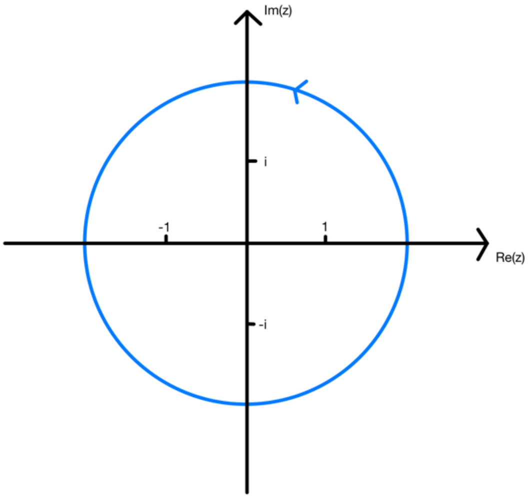

Fig.2 Example $c(t) = 2e^{it}$

By the definition of isolated singular point, (1,0), (0, i), (-1,0), (0, -i) are isolated singular points of $\frac{f(z)}{z^4-1}$ for $c(t) = 2e^{it}$.

4.2.2.2 The isolated singular point for $c(t) = 1 + e^{it}$

Observe the curve of $c(t) = 1 + e^{it}$,(which is showed in figure.3), we notice that (1,0) is inside the curve of $c(t) = 1 + e^{it}$.

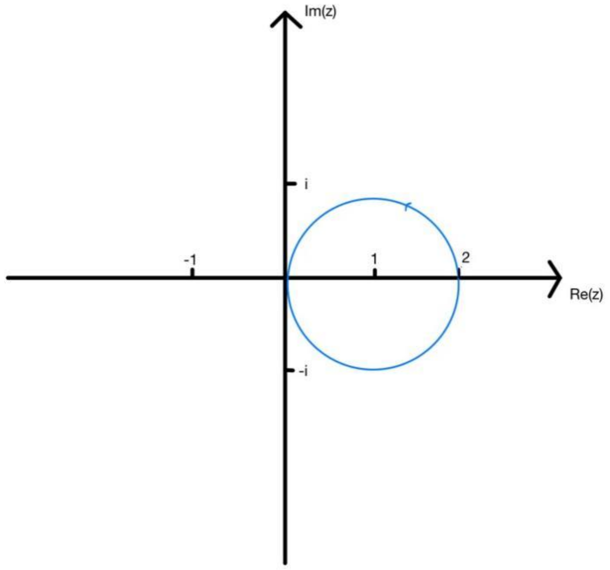

Fig.3 Example $c(t) = 1 + e^{it}$

By the definition of isolated singular point, (1,0), is isolated singular point of $\frac{f(z)}{z^4-1}$ for $c(t) = 1 + e^{it}$.

4.2.2.3 The isolated singular point for $c(t) = \frac{1-i}{2} + \sqrt{2}\, e^{i\,t}$

Observe the curve of $c(t) = \frac{1-i}{2} + \sqrt{2}\, e^{i\,t}$ (which is showed in figure.4), we notice that we notice that (1,0), (0, i), (-1,0) are all inside the curve of $c(t) = \frac{1-i}{2} + \sqrt{2}\, e^{i\,t}$.

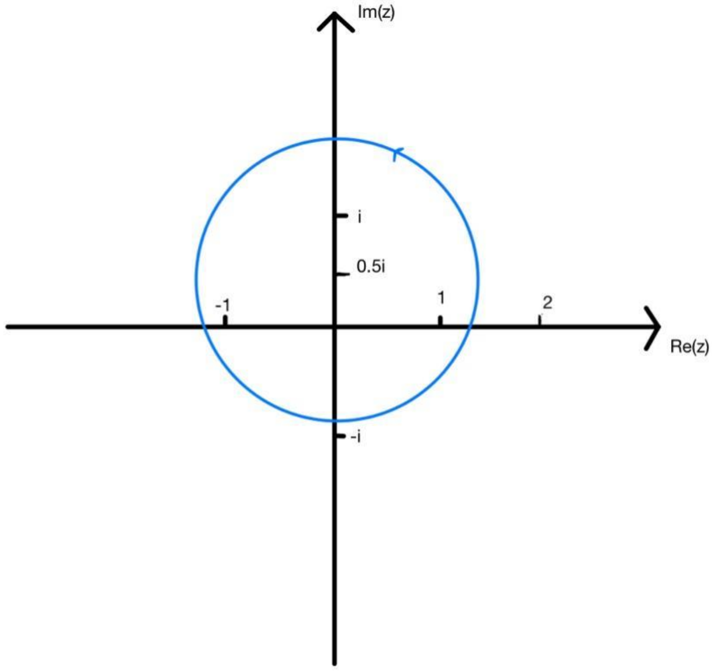

Fig.4 Example $c(t) = \frac{1-i}{2} + \sqrt{2}\, e^{i\,t}$

By the definition of isolated singular point, (1,0), (0, i), (-1,0) are isolated singular points of $\frac{f(z)}{z^4-1}$ for $c(t) = \frac{1-i}{2} + \sqrt{2}\, e^{i\,t}$.

4.2.3 Theorem 1.

For any, $m(z)=\frac{k}{z-z_1}+a$ z is a complex number, the residue in z_1 of this function is k.

4.2.3.1 Proof.

The Laurent expansion (the definition of Laurent expansion can refer to ref [1]) of in z_1 is that

$$\sum_{n=0}^{\infty} a_n(z-z_1)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_1)^n}=\sum_{n=0}^{\infty} a_n(z-z_1)^n+\sum_{n=2}^{\infty}\frac{b_n}{(z-z_1)^n}+\frac{b_1}{z-z_1}=\frac{k}{z-z_1}+a \quad (28)$$

In which,

$$a_n=\frac{1}{2\pi i}\int_c\frac{m(z)dz}{(z-z_1)^{n+1}} \quad (29)$$

$$b_n=\frac{1}{2\pi \mathrm{i}}\int_c\frac{m(z)dz}{(z-z_1)-n+1} \quad (30)$$

So,

$$\mathrm{k}=\mathrm{b}_1=\mathrm{Res}(\mathrm{m},\mathrm{z}_1) \quad (31)$$

4.2.4 The following paths of this method

In this way, for $g(z)$, we have:$Res(g,1)=\mathrm{A}$, $Res(g,i)=\mathrm{B}Res(g,-1)=\mathrm{C}$, $Res(g,-i)=\mathrm{D}$ Then, we try to find out the value of A, B, C, D.

We let $w=\mathrm{i}$. in this way, $w^2=-1$ w^3 = -1, $w^3=-1$

Then, by reduce all the fractions in $\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{z-i}+\frac{D}{z+i}$ into a same denominator, we have:

$$f(z)=10z^3+(2i-2)z^2+2z+2-2i=(A+B+C+D)z^3+(A+Bw+Cw^2+Dw^3)z^2+(A+Bw^2+Cw^4+Dw^6)z+(Aw+Bw^3+Cw^6+Dw^9) \quad (32)$$

So that, $A+B+C+D=10$, $A+Bw+Cw^2+Dw^3=2i+2$, $A+Bw^2+Cw^4+Dw^6=2$,$Aw+Bw^3+Cw^6+Dw^9=2-2i$.

Using these equations, we can form a Fourier matrix (DFT matrix), then find the value of A, B, C, D.(By using the formula(9))

as a result, A = 4, B = 3, C= 2, D = 1.

By the Cauchy residue theorem,

For $c(t)=2e^{it}$

$$\int_{c(t)}\frac{f(z)}{z^4-1}dz=2\pi i(4+3+2+1)=20\pi i$$

For $c(t)=1+e^{it}$

$$\int_{c(t)}\frac{f(z)}{z^4-1}dz=2\pi i(4)=8\pi i$$

For $c(t)=\frac{1-i}{2}+\sqrt{2}\,e^{i\,t}$

$$\int_{c(t)}\frac{f(z)}{z^4-1}dz=2\pi i(4+3+1)=16\pi i$$

4.3. Alternative method.

4.3.1 Theorem 2.

for any$T(z)=\frac{P(z)}{z^4-1}$, z is complex number, the residue at z_p is

$$\frac{z_p}{4}P(z_p) \quad (33)$$

4.3.1.1 Proof.

Assuming that there are two n power series

$$1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \tag{34}$$

$$1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \tag{35}$$

Let

$$(1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\cdots)(1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots\cdots\cdot) = 1 \tag{37}$$

After calculating and summarize, we have:

$$1 = 1 + (b_1 + a_2)z + (b_2 + a_1 b_1 + a_2)z^2 + (b_3 + b_2 a_1 + b_1 a_2 + a_3)z^3 + \cdots \tag{38}$$

In this way, $b_1 = -a_1$,$b_2 = a_1^2 - a_2$, $b_3 = -(a_1^2 - a_2)a_1 - a_1 a_2 - a_3$.

(Because the first coefficient is 1, so all these following coefficients should be zero.)

Then, we apply these conclusions to the task of computing the residue of $\frac{P(z)}{z^4-1}$

Because of the property of isolated singular point, z_p must be one of the 4 roots of $z^4 - 1$. So $z_p^4 = 1$,and $z_p^3 = \frac{1}{z_p}$

The Taylor expansion (The definition of Taylor expansion can refer to ref [1]) of $z^4 - 1$ at z=z_p is that:

(In the following part, we use r_n to represent these coefficients that not relative to prove the theorem)

$$= 0 + 4z_p^3(z - z_p) + \frac{4 \times 3}{2} z_p^2 (z - z_p)^2 + \cdots$$

$$= \frac{4}{z_p}(z - z_p) + r_0 (z - z_p)^2 + r_1 (z - z_p)^3 + \cdots$$

$$= \frac{4}{z_p}(z - z_p)\left(1 + r_0(z - z_p) + r_1(z - z_p)^2 + \cdots\ldots\right) \tag{39}$$

(In the following part, we use r_n to represent these coefficients that not relative to prove the theorem)

In this way,

$$\frac{1}{z^4-1} = \frac{z_p}{4(z-z_p)\left(1+r_0(z-z_p)+r_1(z-z_p)^2+\cdots\ldots\right)} \tag{40}$$

So,

$$\frac{P(z)}{z^4-1} = \frac{z_p}{4(z-z_p)\left(1+r_0(z-z_p)+r_1(z-z_p)^2+\cdots\ldots\right)}\left(P(z_p) + P'(z_p)(z - z_p) + \cdots\right) \tag{41}$$

($P(z_p) + P'(z_p)(z - z_p) + \cdots$ is the Taylor expansion of $P(z)$ at z =z_p.)

Then, after summary,

$$\frac{P(z)}{z^4-1} = \frac{z_p}{4(z-z_p)}\left(P(z_p) + r_0(z - z_p) + r_1(z - z_p)^2 + \cdots\right) \tag{42}$$

The residue of $\frac{P(z)}{z^4-1}$ at z_p. is$\frac{z_p}{4}P(z_p)$.

4.3.2 the following paths of this method

By using this theorem(theorem.2), we can find the residue of $\frac{f(z)}{z^4-1}$, and they are:$Res(g,1) = 4$,$Res(g,i) = 3$,$Res(g,-1) = 2$,$Res(g,-i) = 1$.

By the Cauchy residue theorem,

For $c(t) = 2e^{it}$

$$\int_{c(t)} \frac{f(z)}{z^4 - 1} dz = 2\pi i(4 + 3 + 2 + 1) = 20\pi i$$

For $c(t) = 1 + e^{it}$

$$\int_{c(t)} \frac{f(z)}{z^4 - 1} dz = 2\pi i(4) = 8\pi i$$

For $c(t) = \frac{1-i}{2} + \sqrt{2}\, e^{i\,t}$

$$\int_{c(t)} \frac{f(z)}{z^4 - 1} dz = 2\pi i(4 + 3 + 1) = 16\pi i$$

5. Conclusion

We can now more easily calculate the more complicated and challenging integrals because of the Cauchy's Residue Theorem. Of course, proficiency in the use of specific concepts such as Taylor series, power series and Laurent expansion is a necessity to obtain the required residues. Nevertheless, as long as the residues of some certain limited points calculated by these two methods have been figured out, the value of any integral of the closed counter can be derived according to Cauchy's Residue Theorem. All we need to do is to add up the value of residues that meet the requirements within the simple closed contour, and multiply this sum by 2πi. The content of this theorem can be stated in just a few sentences, but its validity is not to be underestimated; thus, we have developed a heightened interest and veneration for it, especially when it is combined with Fourier Matrix or power series in our examples to show its powerful effectiveness. Additionally, based on many of our reference readings, we have observed that it has uses in a variety of branches of mathematics, including linear algebra and mathematical analysis, probably due to its ease of use. In addition to these elements mentioned above, it also occurs to us that there are times when theorems cannot be applied, such as when the contour does not meet the requirement of being a simple closed one, leading us to more complex occasions. In the future, we are looking forward to using this theorem for more sophisticated and challenging integrals in the area of complex analysis, focusing on its underlying value and more extensive applications, as well as refining our understanding of this theorem.

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