Application of the Residue Theorem to Euler Integral, Gaussian Integral, and Beyond

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Abstract. The paper is divided into three different parts, which use the residue theorem to solve several different integrals, namely, the Euler integral, the Gaussian integral, the Fresnel integral, and so forth. The process of using the resiude theorem to determine these integrals is to first turn the integrals into convenient forms of complex integrals, and then find integral perimeters so that any integral on one of the curves is the required integral, through the drawing observation of the contour to write the original integral into the form of multiple integral. By studying the resiude theorem to solve the problem of complex integrals, it is demonstrated that the resiude theorem is actually a process that makes the calculation easier. These solved integrals have a wide range of applications including the study of the refraction of light, analytics, probability theory, combinatorial mathematics, and unification of the continuous Fourier transform.

Keywords: Residue theorem; Euler integral; Gaussian integral; Fresnel integral.

1. Introduction

The significance of integral is indubitable. Generally, assuming that $f(\mathrm{x})$ is continuous between $[a,b]$ and dividing it into n small parts with $\Delta x = (\mathrm{b}-\mathrm{a})/\mathrm{n}$[1], then the sum of all the small trapezium areas is approximately equal to the real value of the integral if n is large enough. In complex variables, the residue theorem is a powerful tool for calculating the path integral of analytic functions along closed curves, as well as the integral of real functions [2]. The application of residue theorem can transform a finite integral into a path of a complex variables function. Euler integral is widely used in several important regions such as statistic and complex variables [3]. It generalizes two elementary functions: Beta function and Gamma function. Bessel function is generalized from Plath's equation by the method of separation of variables [4]. It plays a very important role in the study of fluctuations and various problems involving potential fields position. The Bessel function is the solution to the Bessel equation [5]. It has a wide range of applications in physics and engineering. Gauss integral has extensive application in the calculation of unification such as probability theory and continuous Fourier transform [6]. It also appears in the definition of error function. Although there is no elementary function error function, the Gaussian integral can be solved by the way of calculus. Gaussian integral, sometimes referred to probability integral, is a Gaussian function [7]. Fresnel integral was firstly applied in the diffraction theory of light [8]. Recently, it is also found in other aspects such as highway and rotation design.

The main formula of Cauchy residue theorem can be defined as the generalization of Cauchy integral theorem and Cauchy integral formula. It states that [9]

$$\int f(z)dz = 2\pi i\,[\mathrm{Res}(f_1, z_1) + \mathrm{Res}(f_1, z_2) + \cdots + \mathrm{Res}(f_1, z_n)]. \tag{1}$$

On the other hand, assuming a function without a middle point z_0, $w$here given a function $h(z) = (z-z_0)^{-1}\varphi(z)$. This assumption makes it easier to do the Taylor expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, -\infty < x < \infty. \tag{2}$$

Now one can easily get $\varphi(z)$=$\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ by using Taylor expansion. When n=0, $a_0 = \varphi(z)$ whereas $a_0 \neq 0$. When n=0, the coefficient of $\frac{1}{z-z_0}$ in series is $\varphi(z)$. By finding extreme value of $z - z_0$, to find the point cannot be analyzed (z_0), so $\mathrm{Res}(f, z_0) = \lim_{z \to z_0}(z - z_0)f(z)$.

2. Main Results

In this section we will use the residue theorem to calculate several important integrals that relate to the Euler integral, Bessel integral, Gaussian integral and Fresnel integral.

2.1. Euler Integral

The Euler integral is perhaps the most widely used of all the known special functions. It is defined as the following integral form [3]

$$\Gamma(\mathrm{z}) = \int_0^\infty s^{a-1}(1+s)^{-1} ds\,. \tag{3}$$

This is a multivalued function, indicating that a single valued branch in the complex plane can be observed when varying the argument from 0 to infinity. In the complex space, the integral can be calculated through the residue theorem,

$$\int_0^\infty z^{a-1}(1+z)^{-1} dz = \int_\varepsilon^{\mathrm{R}} s^{a-1}(1+s)^{-1} ds + \int_R^\varepsilon (se^{2\pi i})^{a-1}(1+s)^{-1} ds$$
$$+ \int_{C_R} z^{a-1}(1+z)^{-1} dz \int_{C_\varepsilon} z^{a-1}(1+z)^{-1} dz$$
$$= -2\pi i \,\mathrm{Res}\left[\frac{s^{a-1}}{1+s}, -1\right], \tag{4}$$

Where contours C_R and C_ε are two different paths with radius of R and ε. This equation looks incredibly tedious and long, but fortunately two of its parts are equal to zero, from these equations. So now just forget about these two equations. Obviously,

$$\int_0^\infty s^{a-1}(1+s)^{-1} ds = \lim_{\varepsilon \to 0, R \to \infty} \int_\varepsilon^{\mathrm{R}} s^{a-1}(1+s)^{-1} ds = \frac{2\pi i}{1 - e^{2\pi\, ia}} \mathrm{Res}\left[\frac{s^{a-1}}{1+s}, -1\right]. \tag{5}$$

Generally, from the residue theorem one can tell that $\mathrm{Res}[f(p)] = \frac{1}{(n-1)!}[(z-p)^n f(z)]^{n-1}$. When $n = 1$, it is reduced to $\mathrm{Res}[f(p)] = \lim_{z \to p}(z-p)f(z)$. By replacing $f(z)$ to $\frac{\varphi(z)}{\psi(z)}$ and letting p be the first-order pole of the denominator, the residue theorem is equivalent to $\mathrm{Res}[f(p)] = \frac{\varphi(z)}{\psi'(z)}$. As a result, we have $\mathrm{Res}\left[\frac{s^{a-1}}{1+s}, -1\right] = (e^{i\pi})^{a-1}$ and the Euler integral

$$I = \frac{2\pi i}{1 - e^{2\pi\, ia}} \mathrm{Res}\left[\frac{z^{a-1}}{1+z}, -1\right] = \frac{2\pi i}{e^{\pi ia}(e^{-\pi ia} - e^{\pi ia})}(e^{i\pi})^{a-1} = \frac{\pi}{\sin a\pi} \tag{6}$$

There are many ways to solve Euler's integral. For different practices, the retention theorem is a good demonstration of how to integrate a complex and difficult to solve the function and finally find the answer and solve. The next example is to the basis of Euler's integral, which is further expanded, to find the relevant different types of integrals, to illustrate the magic use of the retention theorem in

solving the real integral. Although the types are different, with the previous content, it is not difficult to find that the process and method have very similarities.
Next, we consider the following improper integral

$$I=\int_0^{\infty}\frac{dz}{(1+z)z^s} \tag{7}$$

where s is a constant that obey $0<s<1$. Define $f(z)=\frac{1}{(1+z)z^s}$ where z^s denotes a natural branch in the shape of a keyhole contour. By virture of the residue theorem, we have

$$2\pi i\operatorname{Res}(f,-1)=\int_{\varepsilon}^{R}\frac{dx}{(1+x)x^s}+\int_{\Gamma_R}\frac{dz}{(1+z)z^s}+\int_{\varepsilon}^{R}\frac{dx}{(1+x)x^s e^{i2\pi s}}+\int_{\Gamma_{\varepsilon^-}}\frac{dz}{(1+z)(z^{\alpha})_0}. \tag{8}$$

Before carrying on, it is worthwhile to make an estimate of the individual integral. That is, $\int_{\Gamma_{\varepsilon^-}}\frac{dz}{(1+z)(z^{\alpha})_0}\leq\frac{M}{r^{\mathrm{Re}(s)}}\cdot 2\pi r\to 0$ as $r\to 0$ and $\int_{\Gamma_R}\frac{dz}{(1+z)z^s}\leq\frac{M}{r^{1+\mathrm{Re}(s)}}\cdot 2\pi R\to 0$ as $R\to\infty$. Now the required function is already obvious, only the values of the latter two functions need to be estimated. Similarly, we find that another morphological equation is also a region of 0, and finally use the retention theorem to remove ε from 0 and R from infinity. Taken together, we have

$$\left(1-e^{-2\pi i s}\right)\int_0^{\infty}\frac{1}{x^s(1+x)}\,dx=2\pi i\operatorname{Res}(f,-1)=2\pi i(-1)^{-s}=2\pi i e^{-\pi i s}, \tag{9}$$

and thus

$$I=\int_0^{\infty}\frac{1}{x^s(1+x)}\,dx=2\pi i\frac{e^{-\pi i s}}{1-e^{-2\pi i s}}=\frac{\pi}{\sin s\pi}. \tag{10}$$

Unlike the Euler integral mentioned above, this integral has a particularly complex but cumbersome condition, so this integral needs to be discussed in a classification. The general two cases will be discussed, mainly depending on the s value. The charm of the retention theorem is reflected in this. Besides, the use of the retention theorem will turn this integral into the form of adding several integrals, valuation and finding the final answer.

2.2. Bessel Integral

In this subsection we will consider the Bessel-type integral. The integrals are [4,5]

$$l_1=\int_0^{2\pi}e^{cos\alpha}\cos(m\alpha-\sin\alpha)\,d\alpha \tag{11}$$

and

$$l_2=\int_0^{2\pi}e^{cos\alpha}\sin(m\alpha-\sin\alpha)\,d\alpha \tag{12}$$

Generally, it is not easy to evaluate them individually. However, if we put them together and put them into complex variables, the process will be easier. As a result, a new function is created to include both functions so that they can be calculated simultaneously. Let us define $l=l_1-il_2$ and in order to unity all of them into exponential form, we can use the formula $e^{i\alpha}=\cos\alpha+i\sin\alpha$. Hence, we can get

$$l=\int_0^{2\pi}e^{\cos\alpha}\cos(m\alpha-\sin\alpha)-i\,e^{\cos\alpha}sin(m\alpha-sin\alpha)\,d\alpha=\int_0^{2\pi}\frac{e^{\cos\alpha+i\sin\alpha}}{e^{im\alpha}}\,d\alpha. \tag{13}$$

Although this integral is clear enough, we still can't evaluate this integral because we still have triangular on the power side. We can replace the triangular function by a complex variable. Let $z = e^{i\alpha} = \cos\alpha + i\sin\alpha$ and thus

$$l = \int_0^{2\pi} \frac{e^{\cos a + i\sin a}}{e^{im\alpha}} \frac{dz}{ie^{i\alpha}} = \oint \frac{e^z dz}{iz^{m+1}}\,. \tag{14}$$

Next, we get a contour integral and it is clear that there is a residue at $z = 0$. We can use Tylor expansion to find the exact value of the residue. After expansion, notice that all the polynomials equals to zero, we have $\mathrm{Res}[l,0] = \frac{1}{im!}$. From aforementioned content, $\cos\alpha$ corresponds to the real part of the answer while the imaginary part corresponds to $\sin\alpha$. We thus have $l_1 = \mathrm{Re}(l) = \frac{2\pi}{m!}$ and $l_2 = \mathrm{Im}(l) = 0$.

2.3. Gaussian Integral

The residue theorem can be a way to calculate the celebrated Gaussian integral, which states that the integration of the e^{-x^2} from $-\infty$ to ∞ is $\sqrt{\pi}$, or equivalently [6,7],

$$\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}. \tag{15}$$

Historically, there are many different ways to work out this integral. A different approach is adopted by Caldwell, who used the fact that $\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}\, dx$. To conquer this problem, it is necessary to introduce a round contour that is mapped out by $Q = (\frac{1}{2}\sqrt{\pi}, -R)$, $Q = (\frac{1}{2}\sqrt{\pi}, R)$, $S = (-\frac{1}{2}\sqrt{\pi}, R)$, and $T = (-\frac{1}{2}\sqrt{\pi}, -R)$. According to the residue theorem, the sum of the contribution from PQ and ST is $2\int_{-R}^{R} i\exp\left(i\left(\frac{\pi}{4} - y^2\right)\right) dy$. On QS we have $\sin\sqrt{\pi}z > \mathrm{sh}\sqrt{\pi}R$ and the absolute value of $e^{iz^2} = e^{-2Rx}$. From the two equations above, we can get that the contribution of QS should be less than $\int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} \frac{e^{-2Rx}}{\mathrm{sh}\sqrt{\pi}R}\, dx = \frac{1}{R}$. It has a pole at $z = 0$ with the residue of $1/\sqrt{\pi}$, so we can use the Residue theorem by letting R goes to the positive infinity: $\int_0^\infty e^{i\left(\frac{\pi}{4} - Y^2\right)}\, dy = \frac{1}{2}\sqrt{\pi}$. From this we can get $\int_0^\infty \sin\left(\frac{\pi}{4} - y^2\right) dy = 0$ and $\int_0^\infty \cos\left(\frac{\pi}{4} - y^2\right) dy = \frac{\sqrt{\pi}}{2}$. If we expand the two equations above, we can get $\int_0^\infty \cos \mathrm{y}^2\, dy - \int_0^\infty \sin \mathrm{y}^2\, dy = 0$ and $\int_0^\infty \cos y^2\, dy + \int_0^\infty \sin y^2\, dy = \sqrt{\frac{\pi}{2}}$. By solving two equations, we can get:

$$\int_0^\infty \cos y^2 dy = \int_0^\infty \sin y^2 dy = \sqrt{\frac{\pi}{8}}, \tag{16}$$

and also Eq. (15) shows earlier.

2.4. Fresnel Integral

As shown in the last subsection, the Fresnel integral has an intimate relation with the Gaussian integral. Here we provide with an alternative method to calculate the Fresnel integral [8]

$$I = \int_0^\infty \cos x^2\, dx\,. \tag{17}$$

At the first glance, the integral cannot be evaluated by general process. If we consider the complex analysis and the cosine is merely the real part of e^{-x^2}. So we can calculate it by adding an imaginary part and take the real part of the final result, i.e., $f(z)=e^{iz^2}$. Considering the contour integral, we will get a sector disc. It is clear that $\theta=\frac{\pi}{4}$. Let the radius be l_1 and l_2 in anticlockwise direction and the arc be Γ_r. According to Cauchy Integral Theorem：$\int_{l_1} f(z)dz+\int_{l_2} f(z)dz+\int_{\Gamma_r} f(z)dz$=0.

We need to solve them individually and get a relationship between the triangular integral and the final answer. On path l_1, $z=x$, and $\int_{l_1} f(z)dz=\int_0^R e^{ix^2}\,dx=\int_0^R[\cos x^2+i\sin x^2]\,dx$. On path l_2, $z=xe^{\frac{\pi i}{4}}$ $\frac{dz}{dx}=e^{\frac{\pi i}{4}}$, and $\int_{l_2} f(z)dz=\int_0^R e^{x^2e^{\frac{\pi i}{2}}}e^{\frac{\pi i}{4}}\,dx=\int_0^R[\cos x^2-i\sin x^2]\,dx$. On path Γ_r, $\mathrm{z}=\mathrm{xe}^{\mathrm{ix}}$ $\frac{dz}{dx}=ix\mathrm{e}^{\mathrm{ix}}$, and $\int_{\Gamma_r} f(z)dz=\int_0^{\frac{\pi}{4}}\mathrm{e}^{ix^2e^{i2x}}ix\mathrm{e}^{\mathrm{ix}}\,dx=ix\int_0^{\frac{\pi}{4}}e^{ix+ix^2\cos 2x-x^2\sin 2x}\,dx$. As x tends to infinity, the integral tends to zero, $\lim_{x\to\infty}\int_{\Gamma_r} f(z)dz=0$. Taken together,

$$\int_0^{\infty} f(z)dz=e^{\frac{\pi i}{4}}\int_0^R e^{-ix^2}\,dx=e^{\frac{\pi i}{4}}\sqrt{\frac{\pi}{4}}=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)\sqrt{\frac{\pi}{4}}=\sqrt{\frac{\pi}{8}}+\sqrt{\frac{\pi}{8}}i \tag{18}$$

Since $\int_0^{\infty}\cos x^2\,dx$ is the real part of $\int_0^{\infty} f(z)dz$, it is observed that $\int_0^{\infty}\cos x^2 dx=\sqrt{\pi/8}$.

3. Conclusion

In summary, the residue theorem has a wide application in solving many special integrals including Euler integrals, Bessel functions, Gaussian functions, and Fresnel functions [10]. In doing so one need to transform the definite integral into a complex integral. Firstly, in Euler integral the residue theorem is used to rewrite the integral and contour graph is used to solve it. While in Bessel integral, the approach is similar since separating calculations are too complicated to put them into variables. The difference here is that a new function is created to contain both functions. The main purpose here is to use the residue theorem when creating a new function to include these two functions. However, if the integral cannot be calculated after creating a new function, complex variables are used. Finally, for the Gaussian and Fresnel integrals, since the calculations cannot be performed directly, it is necessary to add an imaginary part and use the real part of the result to create a sector diagram and use the Cauchy integral theorem to calculate the path.

In general, when using the residue theorem to calculate special integrals, researchers not only need to consider the application of the residue theorem itself, but also pay attention to the properties of special functions. When it cannot be directly calculated, it is necessary to construct new functions that contain them.

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