On the Calculation of Several Definite Integrals by Residue Theorem

Zixin Huang[1, *, †], Yi Jin[2, †], Moran Yang[3, †] and Yimiao Yuan[4, †]

[1]Westdale Secondary School, Hamilton, Ontario, Canada

[2]Pacific Ridge School, Carlsbad, California, USA

[3]The King's Academy, West Palm Beach, Florida, USA

[4]GuangHua Cambridge International School, Shanghai, China

†These authors contributed equally.

*Corresponding author: zhuang3344@hwdsb.on.ca

Abstract. The Cauchy's residue theorem is one of the most important theorems in complex analysis at all times, and it is demonstrated that using the residue theorem is an easier and faster method to calculate some types of real improper integrals when the targeted integrals are hard or even impossible to deal with by conventional approaches. This paper considers several types of integrals including trigonometric integrals and integrals involving logarithmic function and power function. The general methods for calculating these integrals are presented and typical examples to illustrate how to use the methods are shown. The types of integrals in the paper are useful in many fields and have applications in the engineering and scientific research. In complex analysis, the residue theorem is a powerful tool for calculating the path integrals of analytic functions along closed curves, and can also be used to calculate the integrals of real functions.

Keywords: Residue theorem; Definite integral; Improper integral.

1. Introduction

The definite integrals have a rather wide application in many fields of science. A lot of important physical quantities, free energy and specific heat for instance, are expressed by definite integrals and can be obtained by solving the corresponding integrals [1]. In definite integral, the area of a small space is calculated by imposing constraints, and then it is operated to find the area of the entire space [2]. The definite integrals can be calculated by some conventional methods, but the procedures may be complicated and are not easy to follow [3].

The residue theorem, which is also known as the Cauchy residue theorem [4-6], can help us calculate some definite integrals efficiently. Cauchy considered the difference of integrals along two paths with standard endpoints sandwiching a function pole in the middle, thus forming the residue concept [7]. It is easy to see that, by considering the minus sign as the direction of the path, the difference between the two path integrals can be written as the loop integral of the closed loop formed around the two paths [8]. Therefore, the residue of the function is defined as follow. If $f(z)$ follows the isolated singularity $a \in C$ for the center of the circle $|z-a| = \varepsilon$, then the integral of f which is further divided by $2\pi i$ is called f's residue at point a:

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{2\pi i} \oint_{|z-a|=\varepsilon} f(z)\, dz. \tag{1}$$

Further, the Laurent expansion of the function near $z = a$ is

$$f(z) = \cdots + \frac{c_{-m}}{(z-a)^m} + \cdots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \cdots. \tag{2}$$

It can be proved that the integral $\oint_{|z-a|=R} (z-z_0)^n\, dz = \begin{cases} 2\pi i, n = 1 \\ 0, n \neq 1 \end{cases}$ holds, indicating that $\operatorname*{Res}_{z=a} f(z) = c_{-1}$ [9]. For the resiude theorem, there are also several inferences [10]. For example, If the closed curve C contains multiple poles, then $\oint_C f(z)\,dz = 2\pi i \sum_k \mathrm{Res}_{z=z_k} f$. In addition, If $\mathrm{f} = P/Q$, P and Q are analytic everywhere in the domain, and z_0 is first order pole of Q at z_0, it follows $Res(f; z_0) = \frac{P(z_0)}{Q'(z_0)}$.

Generally, the essence of using the residue theorem to find a definite integral is to first convert the integral into a form convenient for complex integrals and then find an integral encirclement so that the integral on one of the curves is the required definite integral. The integral on the other curves is the required definite integral. More convenient to handle. The specific methods vary with different integral forms; some commonly used methods are briefly described below.

2. Main Results

In what follows, we will apply the residue theorem to calculate several distinct classes of real definite integrals, which include the trigonometric integral, $P(x)/Q(x)$-type integral, $f(x)\ln x$ and $f(x)/x^a$ types integrals.

2.1. Trigonometric Integral

The first sort of integral we shall consider is a trigonometric integral in the range of 0 to 2π. We start by taking the integral

$$I = \int_0^{2\pi} \frac{1}{1 - 2p\cos\theta + p^2}\,d\theta \tag{3}$$

where $0 \le |p| < 1$ is an example to illustrate the method. The first step is to substitute the sines and cosines with complex exponentials by virtue of $z = e^{i\theta} = \cos\theta + i\sin\theta$ so as to $\cos\theta = \frac{z+\frac{1}{z}}{2}$, $\sin\theta = \frac{z-\frac{1}{z}}{2i}$, and $d\theta = \frac{dz}{iz}$. The above integral then becomes $I = \int_{|z|=1} \frac{1}{1-p\left(z+\frac{1}{z}\right)+p^2}\frac{dz}{iz}$. By expanding the denominator $1 - pz + \frac{p}{z} + p^2 = \frac{z-pz^2-p+zp^2}{z} = \frac{z-p-pz(z-p)}{z} = \frac{(z-p)(1-pz)}{z}$, we have $I = \int_{|z|=1} \frac{z}{(z-p)(1-pz)}\frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{1}{(z-p)(1-pz)}dz$. There are two singular points in the fraction, namely $z = p$ or $1/p$. However, only $z = p$ satisfies the condition since $0 \le |p| < 1$. With the residue theorem in mind, we find that the intgeral I is $\frac{2\pi}{1-p^2}$.

The second example is

$$I = \int_0^{2\pi} \frac{\cos\theta}{5 - 3\cos\theta}\,d\theta\,. \tag{4}$$

In contrast to Eq. (3), the difference lies in that there is a cosine in the numerator. To solve it, we can transfer it to $I = \int_{|z|=1} \frac{\frac{z+\frac{1}{z}}{2}}{5-3\left(\frac{z+\frac{1}{z}}{2}\right)}\frac{dz}{iz}$ by a similar procedure. By multiplying the numerator and denominator by $2z$, we can get $I = i\int_{|z|=1} \frac{z^2+1}{z(3z-1)(z-3)}dz$. Similarly, the value of the integral is equal to the residues at $z = 0$ and $1/3$. Hence, the final answer is $I = -2\pi\left(\frac{1}{3} - \frac{5}{12}\right) = \frac{\pi}{6}$.

The third one is

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \cos^2\theta}. \tag{5}$$

Here, there is a quadratic term in the denominator. We transfer it to $\int_{|z|=1} \frac{1}{1+\left(\frac{z+\frac{1}{z}}{2}\right)^2} \frac{dz}{iz}$ $= \int_{|z|=1} \frac{4z}{(z^4+6z^2+1)} \frac{dz}{iz}$, by noticing that $1 + \frac{(z+z^{-1})^2}{4} = 1 + \frac{(z^2+1)^2}{4z^2} = \frac{z^4+4z^2+2z^2+1}{4z^2} = \frac{z^4+6z^2+1}{4z^2}$. Let $z^2 = u$ and the integral becomes $I = \frac{4}{i}\int \frac{du}{u^2+6u+1}$. For this equation, it has one valid first-order pole $u = -3 + 2\sqrt{2}$. Then we can take this into the formula $\mathrm{Res}(f(u), u = -3 + 2\sqrt{2}) = \frac{1}{(-3+2\sqrt{2})+3+2\sqrt{2}} = \frac{1}{4\sqrt{2}}$. Taken together, the integra is $I = 2\pi\mathrm{i} \cdot \frac{4}{i} \cdot \frac{1}{4\sqrt{2}} = \sqrt{2}\pi$.

Finally, the fourth one is

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta, \tag{6}$$

Which has a quadratic term in the numerator. Transfer it to $I = \int_{|z|=1} \frac{\left(\frac{z-\frac{1}{z}}{2i}\right)^2}{a+b\left(\frac{z+\frac{1}{z}}{2}\right)} \frac{dz}{iz}$ and use the same method mentioned above, the numerator can be written in the form of $-\frac{(z^2-1)^2}{4z^2}$. The formula turns out to be $I = \frac{i}{2b}\int_{|z|=1} \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)} dz$ where $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$. The reasonable singular values are $z = 0$ and α. Noticing that the residue at $z = 0$ equals to $-\frac{2a}{b}$ since $\mathrm{Res}(f(z), z = 0) = \frac{(z^2-1)^2}{z^2+\frac{2a}{b}z+1}$ and the residue at $z = \alpha$ equals to $\frac{2\sqrt{a^2-b^2}}{b}$. Since $\mathrm{Res}(f(z), z = \alpha) = \frac{(z^2-1)^2}{z^2(z-\beta)}$, we arrive at the conclusion that that $I = \frac{2\pi}{b^2}\left[a - \sqrt{a^2 - b^2}\right]$.

2.2. $\frac{P(x)}{Q(x)}$-type Integral

In fact, $R(x) = \frac{P(x)}{Q(x)}$ type of integral are utilized in various factors such as selling price, consumer purchasing power, and taxation influence quantity demand and supply, implying that multiple variables control demand and supply. In economics, marginal analysis determines or assesses the change in the value of a function caused by a one-unit increase in one of its variables. Because partial fractions are employed to break down complex rational equations into simpler ones, it plays a major role in determining the production, distribution, and consumption of goods and services.

As a first example, we consider a common integral

$$I = \int_{-\infty}^{+\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx. \tag{7}$$

It is easy to check that $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$ has tow poles on the upper half, namely, $z = i$ and $z = 3i$. Since the residue is $\mathrm{Res}(f(z), i) = -\frac{1+i}{16}$ and $\mathrm{Res}(f(z), 3i) = \frac{3-7i}{48}$, we find that $I = \frac{5}{12}\pi$.

Next, we turn to evaluate a more complicated integral that is of the following form

$$I = \int_{-\infty}^{+\infty} \frac{dx}{(1 + x^2)^3} \tag{8}$$

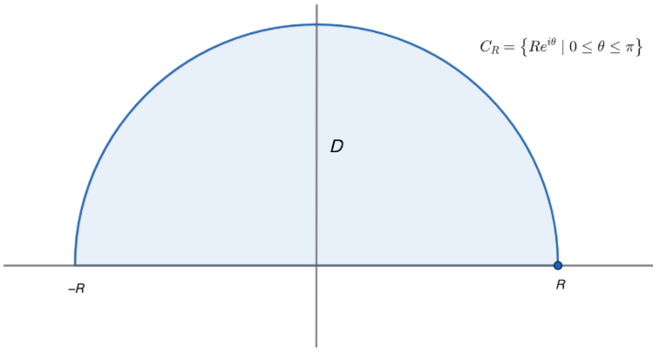

Fig. 1 The general contour for evalulating integral of the type $R(x) = \frac{P(x)}{Q(x)}$.

According to Cauchy's integral theorem, we know that $\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3} = 2\pi i \sum_D \mathrm{Res}[(1+z^2)^{-3}]$ where $z = \pm i$ are third-order poles of integrand. Nevertheless, only $z = i$ is on upper half, as skectched in Fig. 1. According to residue theorem, we can get

$$\mathrm{Res}[(1+z^2)^{-3}] = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{(z-i)^3}{(1+z^2)^3}\right]\Bigg|_{z=i} = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]\Bigg|_{z=i} = -\frac{3}{16}i. \tag{9}$$

As a result, one gets that $I = \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{8}$.

On the other hand, it is also useful to evaluate the following integral

$$I = \int_{-\infty}^{+\infty} \frac{x^{n-1}\sin x}{x^n + 1} dx, n = 1,2,3,\cdots. \tag{10}$$

To solve this problem, we set $R(x) = \frac{x^{n-1}}{x^n+1}$ and the problem above is equivalent to

$$I = \int_{-\infty}^{+\infty} \frac{x^{n-1}\sin x}{x^n + 1} dx = \Im \int_{-\infty}^{+\infty} R(x) e^{ix} dx$$

where $\Im$ represents the imaginary part of a complex number. $R(z)$ has a first-order pole $z_0 = e^{i\frac{\pi}{n}}$, and $\frac{\pi}{n} \le \pi$ since $n \ge 1$. Therefore, $\Im(e^{i\frac{\pi}{n}}) = \sin\frac{\pi}{n} \in (0,1]$. This means that $R(z)$ has a first-order pole z_0 on C_R. According to residue theorem we get $\mathrm{Res}\left[R(z)e^{iz}\right] = \frac{e^{iz_0}}{n}$. Consequnetly, the intgeral in Eq. (10) becomes $\Im \int_{-\infty}^{+\infty} R(x)e^{ix} dx = \Im\left[\frac{2\pi i}{n} e^{iz_0}\right] = \frac{2\pi}{n} e^{-\sin\frac{\pi}{n}} \cos\left(\cos\frac{\pi}{n}\right)$.

Before closing this subsection, we mention that there is also a variation of this type integral, which is of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ikz} dz$. In this case, the degree of $Q(z)$ is higher than that of the $P(z)$, and on the real axis, $Q(z)$ is not equal to 0. A typical example that falls in this classification is

$$I = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2 - 7x + 10} dx\,. \tag{11}$$

In the denominator there are two poles located at $x = 2$ and $x = 5$. According to the residue theorem we have $\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2-7z+10} dz = \pi i \mathrm{Res}\left[f(z)e^{iz}, 2\right] + \mathrm{Res}\left[f(z)e^{iz}, 5\right]$.Taken together, we find that $I = \pi(2\sin 2 - 5\sin 5)$.

2.3. $f(x)\ln x$-type and $f(x)/x^a$ Integrals

In this subsection we consider two related integrals, which is a function f(x) and it multiplies $\ln x$ or $1/x^a$ $(0 < a < 1)$. To evaluate the integral in the form of $\int_0^{\infty} f(x)\ln x\, dx$, we can use the following formula, $\int_0^\infty f(x)\ln x\,dx = -\frac{1}{2}\mathrm{Re}\sum \mathrm{Res}(f(z)\ln^2 z)$, deriving from Cauchy's residue theorem.

As an illustration, we consider the integral

$$I = \int_0^\infty \frac{\ln x}{(x+1)^4}\,dx\,. \tag{12}$$

Letting $f(z) = \frac{1}{(z+1)^4}$, we have $\int_0^\infty \frac{\ln x}{(1+x)^4}dx = -\frac{1}{2}\mathrm{Re}\left[\mathrm{Res}_{z=-1}\frac{\ln^2 z}{(1+z)^4}\right] = -\frac{1}{2}\mathrm{Re}\left(1-\frac{2\pi i}{3}\right) = -\frac{1}{2}$. In addition, we also consider the integral

$$I = \int_0^\infty \frac{\ln x}{x^2+1}dx. \tag{13}$$

Using $f(z) = \frac{1}{z^2+1}$, we have poles $z = \pm i$ and thus $\int_0^\infty \frac{\ln x}{x^2+1}dx = -\frac{1}{2}\mathrm{Re}\left[\mathrm{Res}_{z=i}\frac{\ln^2 z}{x^2+1} + \mathrm{Res}_{z=-i}\frac{\ln^2 z}{x^2+1}\right] = -\frac{1}{2}\mathrm{Re}\left(\frac{i\pi^2}{8}-\frac{9i\pi^2}{8}\right) = 0.$

To evaluate integrals in the form of $\int_0^\infty \frac{f(x)}{x^a}dx$, we first define $f(z)$ with the range of argument of z being $0 < \arg z < 2\pi$, and a path C as shown in the following Fig. 2.

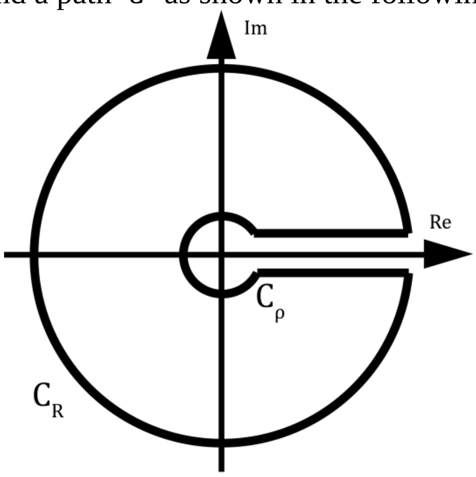

Fig. 2 The contour for evalulating integral.

It is noteworthy that the outer path C_R tends to infinity and the inner path C_ρ tends to 0, because the function is not defined at 0. We then evaluate $f(z)$ along the upper edge and the lower edge and notice that the value of theta differs by 2π. Finally, applying Cauchy's Residue Theorem and simplifying the above expression gives

$$\int_0^\infty \frac{f(x)}{x^a}dx = \frac{2\pi i}{1-e^{-2\pi i a}}\sum \mathrm{Res}_{z=z_k}\left(\frac{f(z)}{z^a}\right) \tag{14}$$

To illustrate it, we consider the integral of the form

$$I = \int_0^\infty \frac{dx}{\sqrt{x}(x^2+4)} \tag{15}$$

Defining $f(z) = \frac{z^{-\frac{1}{2}}}{z^2+4}$ with $0 < \arg z < 2\pi$, integrating it along the path C, and using $z = re^{i\theta}$, we find that $f(z) = \frac{e^{-\frac{1}{2}\log z}}{z^2+4} = \frac{e^{-\frac{1}{2}(\ln r + i\theta)}}{r^2e^{2i\theta}+4}$. On the upper edge, $f(z) = \frac{e^{-\frac{1}{2}(\ln r + 0i)}}{r^2+4} = \frac{r^{-\frac{1}{2}}}{r^2+4}$; while on the lower edge, $f(z) = \frac{e^{-\frac{1}{2}(\ln r + 2i\pi)}}{r^2+4} = \frac{r^{-\frac{1}{2}}e^{-i\pi}}{r^2+4}$. By the residue theorem,

$$\int_{\rho}^{R}\frac{r^{-\frac{1}{2}}}{r^2+4}dr+\int_{C_R}f(z)dz+\int_{\rho}^{R}\frac{r^{-\frac{1}{2}}e^{-i\pi}}{r^2+4}dr+\int_{C_\rho}f(z)dz=2\pi i\left[\operatorname*{Res}_{z=2i}f(z)+\operatorname*{Res}_{z=-2i}f(z)\right]. \quad (16)$$

Finally, the integral under consideration is

$$\int_{0}^{\infty}\frac{1}{\sqrt{r}(r^2+4)}dr=i\pi\left[\operatorname*{Res}_{z=2i}f(z)+\operatorname*{Res}_{z=-2i}f(z)\right]=i\pi\left(\frac{1}{4i\sqrt{2i}}-\frac{1}{4i\sqrt{-2i}}\right)=\frac{\pi}{4}. \quad (17)$$

3. Conclusion

This paper mainly introduces the concept of the residue theorem, calculation method and its application in different integrals. The four kinds of integrals studied have a wide range of applications in economics, chemistry, and physics, including the calculation of demand and supply and selling prices in the market, as well as applications to some atomic problems. As an important method to solve these kinds of integrals, the residue theorem plays a very significant role in solving these integrals. By using the residue theorem, some calculation of periodic integrals and integrals of a large range can be solved, and integral problems can dealt fasterly and more effectively. After a certain period of study and research, a certain amount of knowledge about the residue theorem and its application have been accumulated. After finding some representative problems and re-solving them, a deeper understanding of the residue theorem is showed. In addition, we believe that the research results can provide some ideas for beginners to use the residue theorem to calculate definite integrals, provide a reference for relevant researchers to solve practical problems, and also provide a certain direction for those who have a certain foundation and want to have a deeper understanding of this aspect. However, there are also many unexplored aspects that remain to be further explored.

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